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P H I L O S O P H I C A L.

BY CHARLES HUTTON, LL.D.

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M.DCC.LXXXVI.



TO
HIS GRACE
CHARLES,

DUKE OF RICHMOND, LENNOX, AND AUBIGNY, &c. &c. &c.

MASTER GENERAL OF THE ORDNANCE,

(UNDER WHOSE AUSPICES THE EXPERIMENTS IN GUNNERY WERE MADE)

THESE TRACTS
ARE RESPECTFULLY INSCRIBED, BY

HIS GRACE'S

MOST HUMBLE AND

MOST OBEDIENT SERVANT,

THE AUTHOR.

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P R E F A C E.

THE Author presumes to lay the following Tracts before the Public with the greater confidence, as he hopes these productions of his leisure will be found to bear a due relation to the engagements of his official duty.

The preference given of late, even among professed philosophers, to studies of a less abstract kind, has too frequently diverted the pursuits of mathematicians into paths less suited to their talents, from the desire of a vain and fleeting popularity, instead of the more laudable ambition of making real improvements in the sciences which they had professed to cultivate. The humble consciousness which the author has ever entertained of the limits of his own abilities, has, he hopes, preserved him from this common and pernicious vanity. However solicitous to extend and diversify his own acquirements, he can only hope to add, and that a little, to the public stock of knowledge, in those parts of science to which his early habits,

habits, and subsequent occupations, have led him peculiarly to consecrate his studies.

The very honourable distinction paid to the author by the Royal Society, for his former experiments in gunnery, as well as their general indulgence to his attempts in other mathematical subjects, would perhaps have given an obvious destination to these papers, had he not thought their publication in a collective form, better adapted, from the connection of their subject, to extend their utility.

The first six tracts in this volume will be found to have an obvious connection in respect to their subject; having all of them a tendency either to illustrate the history, or improve the theory of that species of mathematical quantities called Series. The particular subjects of these tracts are sufficiently discussed in the introduction to each of them, respectively, to render any previous detail unnecessary in this place. The author hopes, however, they will be thought to have some claim to the merit of invention; and that their utility will be readily recognized by those who are conversant in these subjects.

The seventh and eighth tracts are rather detached in their nature, relating to subjects purely geometrical. It is hoped, however, that the former of them, being an investigation of some new and curious properties of the sphere and cone, which have always been a fruitful and favourite source of exercise to geometers, will be both acceptable and useful

to those who are engaged in similar speculations. The latter problem, concerning the geometrical division of a circle, having hitherto been deemed impracticable, the solution of it is here given, it is presumed, for the first time.

The largest, and, in the author's opinion, the most important of these tracts, is the ninth, or last; the main purpose of which is altogether practical, though founded in a very subtle and complex theory.

Though the late excellent Mr. Robins first shewed the importance of this theory, and invented a very curious mechanical apparatus for the experiments which he made to verify it, the author is persuaded that none have been hitherto made with cannon balls so completely, as those here related and described; and that these are the first from which the *Data* for determining the resistance of the medium can be accurately derived. It has been the author's great object, next to the accuracy of the experiments, and the full and precise description of them, to simplify the theorems deduced from them, or from the theory itself; of which an example may be found in the new rule given for the velocity of the ball. And it is presumed that the table of the corresponding *Data*, namely, of the Dimensions and Elevation of the gun, and the Range, Velocity and Time of flight of the ball, is now so accurately framed, and so perspicuous, that the several cases of gunnery may be very certainly and easily referred to it; and rules of practice, adapted to the common purposes of the artillerist, may be very readily formed upon these principles.



[First A.]

THE NATURE AND VALUE

any communication that might, perhaps, be made concerning the idea

unrelated to the nature and value of infinite series.

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M A T H E M A T I C A L

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T R A C T I.

A Dissertation on the Nature and Value of Infinite Series.

I. **A**BOUT five years since I discovered a very general and easy method of valuing series whose terms are alternately positive and negative, which equally applies to such series, whether they be converging, or diverging, or their terms all equal; together with several other properties relating to certain series: and as we shall have occasion to deliver some of those matters in the course of these tracts, I shall take this opportunity of premising a few ideas and remarks on the nature and valuation of some of the classes of series which form the object of those communications. This is done with a view to obviate

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any misconceptions that might, perhaps, be made concerning the idea annexed to the term *value* of such series in those intended tracts, and the sense in which it is there always to be understood; which is the more necessary, as many controversies have been warmly agitated concerning these matters, not only of late, by some of our own countrymen, but also by others among the ablest mathematicians in Europe, at different periods in the course of the present century; and all this, it seems, through the want of specifying in what sense the term *value* or *sum* was to be understood in their dissertations. And in this discourse, I shall follow, in a great measure, the sentiments and manner of the late famous L. Euler, contained in a similar memoir of his in the fifth volume of the New Petersburg Commentaries, adding and intermixing here and there other remarks and observations of my own.

2. By a converging series, I mean such a one whose terms continually decrease; and by a diverging series, that whose terms continually increase. So that a series whose terms neither increase nor decrease, but are all equal, as they neither converge nor diverge, may be called a neutral series, as $a - a + a - a + \&c.$ Now converging series, being supposed infinitely continued, may have their terms decreasing to 0 as a limit, as the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \&c.$ or only decreasing to some finite magnitude as a limit, as the series $\frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \&c.$ which tends continually to 1 as a limit. So in like manner, diverging series may have their terms tending to a limit that is either finite or infinitely great; thus the terms $1 - 2 + 3 - 4 + \&c.$ diverge to infinity, but the diverging terms $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \&c.$ only to the finite magnitude 1. Hence then, as the ultimate terms of series which do not converge to 0, by supposing them continued *in infinitum*, may be either finite or infinite, there will be two kinds of such series, each of which will be farther divided into two species, according as the terms shall either be all affected with the same sign, or have alternately the signs + and -. We shall, therefore, have altogether four species of series which do not converge to 0, an example of each of which may be as here follows:

I.



$$\begin{array}{ll}
 \text{I.} & \left\{ \begin{array}{l} 1 + 1 + 1 + 1 + 1 + 1 + \&c. \\ \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7} + \&c. \end{array} \right. \\
 \text{II.} & \left\{ \begin{array}{l} 1 - 1 + 1 - 1 + 1 - 1 + \&c. \\ \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6} - \frac{6}{7} + \&c. \end{array} \right. \\
 \text{III.} & \left\{ \begin{array}{l} 1 + 2 + 3 + 4 + 5 + 6 + \&c. \\ 1 + 2 + 4 + 8 + 16 + 32 + \&c. \end{array} \right. \\
 \text{IV.} & \left\{ \begin{array}{l} 1 - 2 + 3 - 4 + 5 - 6 + \&c. \\ 1 - 2 + 4 - 8 + 16 - 32 + \&c. \end{array} \right.
 \end{array}$$

3. Now concerning the sums of these species of series, there have been great diffensions among mathematicians; some affirming that they can be expressed by a certain sum, while others deny it. In the first place, however, it is evident that the sums of such series as come under the first of these species, will be really infinitely great, since by actually collecting the terms, we can arrive at a sum greater than any proposed number whatever: and hence there can be no doubt but that the sums of this species of series may be exhibited by expressions of this kind $\frac{a}{0}$. It is concerning the other species, therefore, that mathematicians have chiefly differed; and the arguments which both sides allege in defence of their opinions, have been endued with such force, that neither party could hitherto be brought to yield to the other.

4.* As to the second species, the famous Leibnitz was one of the first who treated of this series $1 - 1 + 1 - 1 + 1 - 1 + \&c.$ and he concluded the sum of it to $= \frac{1}{2}$, relying upon the following cogent reasons. And first, that this series arises by resolving the fraction $\frac{1}{1+a}$ into the series $1 - a + a^2 - a^3 + a^4 - a^5 + \&c.$ by continual division in the usual way, and taking the value of a equal to unity. Secondly, for more confirmation, and for persuading such as are not accustomed to calculations, he reasons in the following manner: If the series terminate any where, and if the number of the terms be even, then its value will be $= 0$;

but if the number of terms be odd, the value of the series will be $= 1$: but because the series proceeds *in infinitum*, and that the number of the terms cannot be reckoned either odd or even, we may conclude that the sum is neither $= 0$, nor $= 1$, but that it must obtain a certain middle value, equidifferent from both, and which is therefore $= \frac{1}{2}$. And thus, he adds, nature adheres to the universal law of justice, giving no partial preference to either side.

5. Against these arguments the adverse party make use of such objections as the following. First, that the fraction $\frac{1}{1+a}$ is not equal to the infinite series $1 - a + a^2 - a^3 + \&c.$ unless a be a fraction less than unity. For if the division be any where broken off, and the quotient of the remainder be added, the cause of the paralogism will be manifest; for we shall then have $\frac{1}{1+a} = 1 - a + a^2 - a^3 + \dots \pm a^n \mp \frac{a^{n+1}}{1+a}$; and that although the number n should be made infinite, yet the supplemental fraction $\mp \frac{a^{n+1}}{1+a}$ ought not to be omitted, unless it should become evanescent, which happens only in those cases in which a is less than 1, and the terms of the series converge to 0. But that in other cases there ought always to be included this kind of supplement $\mp \frac{a^{n+1}}{1+a}$; and although it be affected with the dubious sign $\mp$, namely $-$ or $+$ according as n shall be an even or an odd number, yet if n be infinite, it may not therefore be omitted, under the pretence that an infinite number is neither odd nor even, and that there is no reason why the one sign should be used rather than the other; for it is absurd to suppose that there can be any integer number, even although it be infinite, which is neither odd nor even.

6. But this objection is rejected by those who attribute determinate sums to diverging series, because it considers an infinite number as a determinate number, and therefore either odd or even, when it is really inde-

indeterminate. For that it is contrary to the very idea of a series, said to proceed *in infinitum*, to conceive any term of it as the last, although infinite: and that therefore the objection above-mentioned, of the supplement to be added or subtracted, naturally falls of itself. Therefore, since an infinite series never terminates, we never can arrive at the place where that supplement must be joined; and therefore that the supplement not only may, but indeed ought to be neglected, because there is no place found for it.

And these arguments, adduced either for or against the sums of such series as above, hold also in the fourth species, which is not otherwise embarrassed with any further doubts peculiar to itself.

7. But those who dispute against the sums of such series, think they have the firmest hold in the third species. For although the terms of these series continually increase, and that, by actually collecting the terms, we can arrive at a sum greater than any assignable number, which is the very definition of infinity; yet the patrons of the sums are forced to admit, in this species, series whose sums are not only finite, but even negative, or less than nothing. For since the fraction $\frac{1}{1-a}$, by evolving it by division, becomes $1 + a + a^2 + a^3 + a^4 + \&c.$ we should have

$$\frac{1}{1-2} = -1 = 1 + 2 + 4 + 8 + 16 + \&c.$$

$$\frac{1}{1-3} = -\frac{1}{2} = 1 + 3 + 9 + 27 + 81 + \&c.$$

which their adversaries, not undeservedly, hold to be absurd, since by the addition of affirmative numbers, we can never obtain a negative sum; and hence they urge that there is the greater necessity for including the before-mentioned supplement additive, since by taking it in, it is evident that $-1 = 1 + 2 + 4 + 8 \dots 2^n + \frac{2^{n+1}}{1-2}$, although n should be an infinite number.

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8. The defenders therefore of the sums of such series, in order to reconcile this striking paradox, more subtle perhaps than true, make a distinction between negative quantities; for they argue that while some are less than nothing, there are others greater than infinite, or above infinity. Namely, that the one value of -1 ought to be understood, when it is conceived to arise from the subtraction of a greater number $a + 1$ from a less a ; but the other value, when it is found equal to the series $1 + 2 + 4 + 8 + \&c.$ and arising from the division of the number 1 by -1 ; for that in the former case it is less than nothing, but in the latter greater than infinite. For the more confirmation, they bring this example of fractions

$$\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{1}, \frac{1}{0}, \frac{1}{-1}, \frac{1}{-2}, \frac{1}{-3}, \&c.$$

which, evidently increasing in the leading terms, it is inferred will continually increase; and hence they conclude that $\frac{1}{-1}$ is greater than $\frac{1}{0}$, and $\frac{1}{-2}$ greater than $\frac{1}{-1}$, and so on: and therefore as $\frac{1}{-1}$ is expressed by -1 , and $\frac{1}{0}$ by ∞ or infinity, -1 will be greater than ∞ , and much more will $-\frac{1}{2}$ be greater than ∞ . And thus they ingeniously enough repelled that apparent absurdity by itself.

9. But although this distinction seemed to be ingeniously devised, it gave but little satisfaction to the adversaries; and besides, it seemed to affect the truth of the rules of algebra. For if the two values of -1 , namely $1 - 2$ and $\frac{1}{-1}$, be really different from each other, as we may not confound them, the certainty and the use of the rules, which we follow in making calculations, would be quite done away; which would be a greater absurdity than that for whose sake the distinction was devised: but if $1 - 2 = \frac{1}{-1}$, as the rules of algebra require, for by multiplication $-1 \times 1 - 2 = -1 + 2 = 1$, the matter in debate is not settled; since the quantity -1 , to which the series $1 + 2 + 4 + 8 + \&c.$ is made equal, is less than nothing, and therefore the same difficulty still remains. In the mean time however, it seems but agreeable to truth, to say that the same quantities which are below nothing, may be taken

taken as above infinite. For we know, not only from algebra, but from geometry also, that there are two ways, by which quantities pass from positive to negative, the one through the cypher or nothing, and the other through infinity: and besides that quantities, either by increasing or decreasing from the cypher, return again, and revert to the same term 0; so that quantities more than infinite are the same with quantities less than nothing, like as quantities less than infinite agree with quantities greater than nothing.

10. But, farther, those who deny the truth of the sums that have been assigned to diverging series, not only omit to assign other values for the sums, but even set themselves utterly to oppose all sums belonging to such series, as things merely imaginary. For a converging series, as suppose this $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c.$ will admit of a sum $= 2$, because the more terms of this series we actually add, the nearer we come to the number 2: but in diverging series the case is quite different; for the more terms we add, the more do the sums which are produced differ from one another, neither do they ever tend to any certain determinate value. Hence they conclude that no idea of a sum can be applied to diverging series, and that the labour of those persons who employ themselves in investigating the sums of such series, is manifestly useless, and indeed contrary to the very principles of analysis.

11. But notwithstanding this seemingly real difference, yet neither party could ever convict the other of any error, whenever the use of series of this kind has occurred in analysis; and for this good reason, that neither party is in an error, but that the whole difference consists in words only. For if in any calculation I arrive at this series $1 - 1 + 1 - 1 + \&c.$ and that I substitute $\frac{1}{2}$ instead of it; I shall surely not thereby commit any error; which however I should certainly incur if I substitute any other number instead of that series; and hence there remains no doubt but that the series $1 - 1 + 1 - 1 + \&c.$ and the fraction

fraction $\frac{1}{2}$, are equivalent quantities, and that the one may always be substituted instead of the other without error. So that the whole matter in dispute seems to be reduced to this only, namely, whether the fraction $\frac{1}{2}$ can be properly called the *sum* of the series $1 - 1 + 1 - 1 + \&c.$ Now if any persons should obstinately deny this, since they will not however venture to deny the fraction to be equivalent to the series, it is greatly to be feared they will fall into mere quarrelling about words.

12. But perhaps the whole dispute will easily be compromised, by carefully attending to what follows. Whenever, in analysis, we arrive at a complex function or expression, either fractional or transcendental; it is usual to convert it into a convenient series, to which the remaining calculus may be more easily applied. And hence the occasion and rise of infinite series. So far only then do infinite series take place in analytics, as they arise from the evolution of some finite expression; and therefore, instead of an infinite series, in any calculus, we may substitute that formula, from whose evolution it arose. And hence, for performing calculations with more ease or more benefit, like as rules are usually given for converting into infinite series such finite expressions as are endued with less proper forms; so, on the other hand, those rules are to be esteemed not less useful by the help of which we may investigate the finite expression from which a proposed infinite series would result, if that finite expression should be evolved by the proper rules: and since this expression may always, without error, be substituted instead of the infinite series, they must necessarily be of the same value: and hence no infinite series can be proposed, but a finite expression may, at the same time, be conceived as equivalent to it.

13. If therefore, we only so far change the received notion of a sum as to say, that the sum of any series, is the finite expression by the evolution of which that series may be produced, all the difficulties, which

which have been agitated on both sides, vanish of themselves. For, first, that expression by whose evolution a converging series is produced, exhibits at the same time its sum, in the common acceptation of the term: neither, if the series should be divergent, could the investigation be deemed at all more absurd, or less proper, namely, the searching out a finite expression which, being evolved according to the rules of algebra, shall produce that series. And since that expression may be substituted in the calculation instead of this series, there can be no doubt but that it is equal to it. Which being the case, we need not necessarily deviate from the usual mode of speaking, but might be permitted to call that expression also the *sum*, which is *equal* to any series whatever, provided however, that, in series whose terms do not converge to 0, we do not connect that notion with this idea of a sum, namely, that the more terms of the series are actually collected, the nearer we must approach to the value of the sum.

14. But if any person shall still think it improper to apply the term sum, to the finite expressions by whose evolution all series in general are produced; it will make no difference in the nature of the thing; and instead of the word sum, for such finite expression, he may use the term value; or perhaps the term *radix* would be as proper as any other that could be employed for this purpose, as the series may justly be considered as issuing or growing out of it, like as a plant springs from its root, or from its seed. The choice of terms being in a great measure arbitrary, every person is at liberty to employ them in whatever sense he may think fit, or proper for the purpose in hand; provided always that he fix and determine the sense in which he understands or employs them. And as I consider any series, and the finite expression by whose evolution that series may be produced, as no more than two different ways of expressing one and the same thing, whether that finite expression be called the sum, or value, or radix of the series; so in the following paper, and in some others which may perhaps hereafter

be produced, it is in this sense I desire to be understood when searching out the value of series, namely, that the object of my enquiry, is the radix by whose evolution the series may be produced, or else an approximation to the value of it in decimal numbers, &c.

Royal Military Acad.
Woolwich, May 24, 1785.



A new Method for the Valuation of Numeral Infinite Series, whose Terms are alternately (+) Plus and (−) Minus; by taking continual Arithmetical Means between the Successive Sums, and their Means.

Article 1. **T**HE remarkable difference between the facility which mathematicians have found in their endeavours to determine the values of infinite series whose terms are alternately affirmative and negative, and the difficulty of doing the same thing with respect to those series whose terms are all affirmative, is one of those striking appearances in science which we can hardly persuade ourselves is true, even after we have seen many proofs of it, and which serve to put us ever after on our guard not to trust to our first notions, or conjectures, on these subjects, till we have brought them to the test of demonstration. For, at first sight it is very natural to imagine, that those infinite series whose terms are all affirmative, or added to the first term, must be much simpler in their nature, and much easier to be summed, than those whose terms are alternately affirmative and negative; which, nevertheless, we find, upon examination, to be directly contrary to the truth; the methods of finding the sums of the latter series being numerous and easy, and also very general, whereas those that have been hitherto discovered for the summation of the former series, are few and difficult, and confined to series whose terms are generated from each other according to some particular laws, instead of extending, as the other methods do, to all sorts of series, whose terms are connected together by addition, by whatever law their terms are formed. Of this remarkable difference between these two sorts of series, the new method of finding the sums of those whose terms are

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alternately

alternately positive and negative, which is the subject of the present paper, will afford us a striking instance, as it possesses the happy qualities of simplicity, ease, perspicuity, and universality in a great degree; and yet, as the essence of it consists in the alternation of the signs $+$ and $-$, by which the terms are connected with the first term, it is of no use in the summation of those other series whose terms are all connected with each other by the sign $+$.

2. This method, so easy and general, is, in short, simply this: beginning at the first term a of the series $a - b + c - d + e - f + \&c.$ which is to be summed, compute several successive values of it, by taking in successively more and more terms, one term being taken in at a time; so that the first value of the series shall be its first term a , (or even 0 or nothing may begin the series of sums); the next value shall be its first two terms $a - b$, reduced to one number; its next value shall be the first three terms $a - b + c$, reduced to one number; its next value shall be the first four terms $a - b + c - d$, reduced also to one number; and so on. This, it is evident, may be done by means of the easy arithmetical operation of addition and subtraction. And then, having found a sufficient number of successive values of the series, more or less as the case may require, interpose between these values a set of arithmetical mean quantities or proportionals; and between these arithmetical means interpose a second set of arithmetical mean quantities; and between those arithmetical means of the second set, interpose a third set of arithmetical mean quantities; and so on as far as you please. By this process we soon find either the true value of the series proposed, when it has a determinate rational value, or otherwise we obtain several sets of values approximating nearer and nearer to the sum of the series, both in the columns and in the lines, either horizontal or obliquely descending or ascending; namely, both of the several sets of means themselves, and the sets or series formed of any of their corresponding terms, as of all their first terms, of their second terms,
of

of their third terms, &c. or of their last terms, of their penultimate terms, of their antepenultimate terms, &c. and if between any of these latter sets, consisting of the like or corresponding terms of the former sets of arithmetical means, we again interpose new sets of arithmetical means, as we did at first with the successive sums, we shall obtain other sets of approximating terms, having the same properties as the former. And thus we may repeat the process as often as we please, which will be found very useful in the more difficult diverging series, as we shall see hereafter. For this method, being derived only from the circumstance of the alternation of the signs of the terms (+ and -), it is therefore not confined to converging series alone, but is equally applicable both to diverging series, and to *neutral* series, (by which last name I shall take the liberty to distinguish those series whose terms are all of the same constant magnitude); namely, the application is equally the same for all the three following sorts of series, viz.

Converging, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \&c.$

Diverging, $1 - 2 + 3 - 4 + 5 - 6 + \&c.$

Neutral, $1 - 1 + 1 - 1 + 1 - 1 + \&c.$

As is demonstrated in what follows, and exemplified in a variety of instances.

It must be noted, however, that by the value of the series, I always mean such *radix*, or finite expression, as, by evolution, would produce the series in question; according to the sense I have stated in the former paper, on this subject; or an approximate value of such radix; and which radix, as it may be substituted instead of the series in any operation, I call the value of the series.

3. It is an obvious and well-known property of infinite series, with alternate signs, that when we seek their value by collecting their terms one after another, we obtain a series of successive sums, which approach continually nearer and nearer to the true value of the proposed series, when it is a converging one, or one whose terms always decrease
by

by some regular law ; but in a diverging series, or one whose terms as continually increase, those successive sums diverge always more and more from the true value of the series. And from the circumstance of the alternate change of the signs, it is also a property of those successive sums, that when the last term which is included in the collection, is a positive one, then the sum obtained is too great, or exceeds the truth ; but when the last collected term is negative, then the sum is too little, or below the truth. So that, in both the converging and diverging series, the first term alone, being positive, exceeds the truth ; the second sum, or the sum of the first two terms, is below the truth ; the third sum, or the sum of three terms, is above the truth ; the fourth sum, or the sum of four terms, is below the truth ; and so on ; the sum of any even number of terms being below the true value of the series, and the sum of any odd number, above it. All which is generally known, and evident from the nature and form of the series. So, of the series $a - b + c - d + e - f + \&c.$ the first sum a is too great ; the second sum $a - b$ too little ; the third sum $a - b + c$ too great ; and so on as in the following table, where s is the true value of the series, and o is placed before the collected sums, to compleat the series, being the value when no terms are included :

Successive sums.

s is greater than	o
s is less than	a
s is greater than	$a - b$
s is less than	$a - b + c$
s is greater than	$a - b + c - d$
s is less than	$a - b + c - d + e$
$\&c.$	$\&c.$

4. Hence the value of every alternate series s , is positive, and less than the first term a , the series being always supposed to begin with a positive term a ; and consequently if the signs of all the terms be changed,

changed, or if the series begin with a negative term, the value s will still be the same, but negative, or the sign of the sum will be changed, and the value become $-s = -a + b - c + d - \&c.$ Also, because the successive sums, in a converging series, always approach nearer and nearer to the true value, while they recede always farther and farther from it in a diverging series; it follows that, in a neutral series, $a - a + a - a + \&c.$ which holds a middle place between the two former, the successive sums $0, a, 0, a, 0, a \&c.$ will neither converge nor diverge, but will be always at the same distance from the value of the proposed series $a - a + a - a + \&c.$ and consequently that value will always be $= \frac{1}{2}a$, which holds every where the middle place between 0 and a .

5. Now, with respect to a converging series, $a - b + c - d + \&c.$ because 0 is below, and a above s the value of the series, but a nearer than 0 to the value s , it follows that s lies between a and $\frac{1}{2}a$, and that $\frac{1}{2}a$ is less than s , and so nearer to s than 0 is. In like manner, because a is above, and $a - b$ below the value s , but $a - b$ nearer to that value than a is, it follows that s lies between a and $a - b$, and that the arithmetical mean $a - \frac{1}{2}b$ is something above the value of s , but nearer to that value than a is. And thus, the same reasoning holding in every following pair of successive sums, the arithmetical means between them will form another series of terms, which are, like those sums, alternately less and greater than the value of the proposed series, but approximating nearer to that value than the several successive sums do, as every term of those means is nearer to the value s than the corresponding preceding term in the sums is. And like as the successive sums form a progression approaching always nearer and nearer to the value of the series, so in like manner their arithmetical means form another progression coming nearer and nearer to the same value, and each term of the progression of means nearer than each term of the successive sums. Hence then we have the two following series,

series, namely, of successive sums and their arithmetical means, in which each step approaches nearer to the value of s than the former, the latter progression being however nearer than the former, and the terms or steps of each alternately below and above the value s of the series $a - b + c - d + \&c.$

Succeffive sums	Arithmetical means
> 0	$> \frac{1}{2}a$
$< a$	$< a - \frac{1}{2}b$
$> a - b$	$> a - b + \frac{1}{2}c$
$< a - b + c$	$< a - b + c - \frac{1}{2}d$
$> a - b + c - d$	$> a - b + c - d + \frac{1}{2}e$
$< a - b + c - d + e$	$< a - b + c - d + e - \frac{1}{2}f$
$\&c.$	$\&c.$

where the mark $>$, placed before any step, signifies that it is too little, or below the value s of the converging series $a - b + c - d + \&c.$ and the mark $<$ signifies the contrary, or too great. And hence $\frac{1}{2}a$, or half the first term of such a converging series, is less than s the value of the series.

6. And since these two progressions possess the same properties, but only the terms of the latter nearer to the truth than the former; for the very same reasons as before, the means between the terms of these first arithmetical means, will form a third progression, whose terms will approach still nearer to the value of s than the second progression, or the first means; and the means of these second means will approach nearer than the said second means do; and so on continually, every succeeding order of arithmetical means, approaching nearer to the value of s than the former. So that the following columns of sums and means will be each nearer to the value of s than the former, viz.

Succeffive

	Suc. fums.	1st means.	2d means.	3d means.	4th means.
> 0	$\frac{a}{2}$	$\frac{3a-b}{4}$	$\frac{7a-4b+c}{8}$	$\frac{15a-11b+5c-d}{16}$	
< a	$a-\frac{b}{2}$	$a-\frac{3b-c}{4}$	$a-\frac{7b-4c+d}{8}$	$a-\frac{15b-11c+5d-e}{8}$	
> a-b	$a-b+\frac{c}{2}$	$a-b+\frac{3c-d}{4}$	$a-b+\frac{7c-4d+e}{8}$	$a-b+\frac{15c-11d+5e-f}{8}$	
< a-b+c	$a-b+c-\frac{d}{2}$	$a-b+c-\frac{3d-e}{4}$	$a-b+c-\&c.$	$a-b+\&c.$	
> a-b+c-d	$a-b+\&c.$	$a-b+\&c.$	$a-b+\&c.$	$a-b+\&c.$	
&c.					

Where every column consists of a set of quantities, approaching still nearer and nearer to the value of s , the terms of each column being alternately below and above that value, and each succeeding column approaching nearer than the preceding one. Also every line, formed of all the first terms, all the second terms, all the third terms, &c. of the columns, forms also a progression whose terms continually approximate to the value of s , and each line nearer or quicker than the former; but differing from the columns, or vertical progressions, in this, namely, that whereas the terms in the columns are alternately below and above the value of s , those in each line are all on one side of the value s , namely, either all below or all above it; and the lines alternately too little and too great, namely all the expressions in the first line too little, all those in the second line too great, those in the third line too little, and so on, every odd line being too little, and every even line too great.

7. Hence the expressions

$$0, \frac{a}{2}, \frac{3a-b}{4}, \frac{7a-4b+c}{8}, \frac{15a-11b+5c-d}{16}, \frac{31a-26b+16c-6d+e}{32}, \&c.$$

are continual approximations to the value s of the converging series $a - b + c - d + e - \&c.$ and are all below the truth. But we can easily express all these several theorems by one general formula. For, since it is evident by the construction, that whilst the denominator in any one of them is some power (2^n) of 2 or $1 + 1$, the numeral co-

efficients of $a, b, c, \&c.$ the terms in the numerator, are found by subtracting all the terms except the last term, one after another, from the said power 2^n or $1+1^n$ which is $= 1+n+n \cdot \frac{n-1}{2} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} + \&c.$ namely the coefficient of a equal to all the terms 2^n minus the first term 1 ; that of b equal to all except the first two terms $1+n$; that of c equal to all except the first three; and so on, till the coefficient of the last term be $= 1$ the last term of the power; it follows that the general expression for the several theorems, or the general approximate value of the converging series $a - b + c - d + \&c.$ will be

$$\frac{2^n - 1}{2^n} a - \frac{2^n - 1 - n}{2^n} b + \frac{2^n - 1 - n - n \cdot \frac{n-1}{2}}{2^n} c - \frac{2^n - 1 - n - n \cdot \frac{n-1}{2} - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}}{2^n} d$$

 $+ \&c.$ continued till the terms vanish and the series break off, n being equal to 0 or any integer number. Or this general formula may be expressed by this series,

$$\frac{1}{2^n} \times : 2^n - 1 \cdot a - A - n \cdot b + B - n \cdot \frac{n-1}{2} \cdot c - C - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot d + \&c.$$

where $A, B, C, \&c.$ denote the coefficients of the several preceding terms. And this expression, which is always too little, is the nearer to the true value of the series $a - b + c - d + \&c.$ as the number n is taken greater: always excepting however those cases in which the theorem is accurately true when n is some certain finite number. Also, with any value of n , the formula is nearer to the truth, as the terms $a, b, c, \&c.$ of the proposed series, are nearer to equality; so that the slower the proposed series converges, the more accurate is the formula, and the sooner does it afford a near value of that series: which is a very favourable circumstance, as it is in cases of very slow convergency that approximating formulæ are chiefly wanted. And, like as the formula approaches nearer to the truth as the terms of the series approach to an equality, so when the terms become quite equal, as in a neutral series, the formula becomes quite accurate, and always gives the same value $\frac{1}{2}a$ for s or the series, whatever integer number be taken for n . And farther, when the proposed series, from being converging, passes through neutrality,

neutrality, when its terms are equal, and becomes diverging, the formula will ſtill hold good, only it will then be alternately too great, and too little as long as the ſeries diverges, as we ſhall preſently ſhew more fully. So that in general the value s of the ſeries $a - b + c - d + \&c.$ whether it be converging, diverging, or neutral, is leſs than the firſt term a ; when the ſeries converges, the value is above $\frac{1}{2}a$; when it diverges, it is below $\frac{1}{2}a$; and when neutral, it is equal to $\frac{1}{2}a$.

8. Take now the ſeries of the firſt terms of the ſeveral orders of arithmetical means, which form the progreſſion of continual approximating formulæ, being each nearer to the value of the ſeries $a - b + c - d + \&c.$ than the former, and place them in a column one under another; then take the differences between every two adjacent formulæ, and place in another column by the ſide of the former, as here below:

Approx. Formulæ.

$$\begin{array}{r}
 a \\
 \hline
 2 \\
 3a - b \\
 \hline
 4 \\
 7a - 4b + c \\
 \hline
 8 \\
 15a - 11b + 5c - d \\
 \hline
 16 \\
 31a - 26b + 16c - 6d + e \\
 \hline
 32 \\
 \&c.
 \end{array}$$

Differences.

$$\begin{array}{r}
 a - b \\
 \hline
 4 \\
 a - 2b + c \\
 \hline
 8 \\
 a - 3b + 3c - d \\
 \hline
 16 \\
 a - 4b + 6c - 4d + e \\
 \hline
 32 \\
 \&c.
 \end{array}$$

From which it appears, that this ſeries of differences conſiſts of the very ſame quantities, which form the firſt terms of all the orders of differences of the terms of the propoſed ſeries $a - b + c - d + \&c.$ when taken as uſual in the differential method. And becauſe the firſt of the above differences added to the firſt formula, gives the ſecond formula; and the ſecond difference added to the ſecond formula, gives the third formula; and ſo on; therefore the firſt formula with all the differences added, will give the laſt formula; conſequently our general formula

$$\frac{1}{2^n} \times : 2^n - 1 . a - A - n . b + B - n . \frac{n-1}{2} . c - C - n \frac{n-1}{2} . \frac{n-2}{3} . d + \&c.$$

D 2

which

which approaches to the value of the series $a - b + c - d + \&c.$ is also equivalent to, or reduces to this form,

$$\frac{a}{2} + \frac{a-b}{4} + \frac{a-2b+c}{8} + \frac{a-3b+3c-d}{16} + \frac{a-4b+6c-4d+e}{32} + \&c.$$

which, it is evident, agrees with the famous differential series. And this coincidence might be sufficient to establish the truth of our method, though we had not given other more direct proof of it. Hence it appears then, that our theorem is of the same degree of accuracy, or convergency, as the differential theorem; but admits of more direct and easy application, as the terms themselves are used, without the previous trouble of taking the several orders of differences. And our method will be rendered general for literal as well as numeral series, by supposing $a, b, c, \&c.$ to represent, not barely the coefficients of the terms, but the whole terms, both the numeral and the literal part of them. However, as the chief use of my method is to obtain the numeral value of series, when a literal series is to be so summed, it is to be made numeral by substituting the numeral values of the letters instead of them. It is farther evident, that we might easily derive our method of arithmetical means from the above differential series, by beginning with it, and receding back to our theorems, by a counter process to that above given.

9. Having, in Art. 5, 6, 7, 8, completed the investigations for the first or converging form of series, the first four articles being introductory to both forms in common; we may now proceed to the diverging form of series, for which we shall find the same method of arithmetical means, and the same general formula, as for the converging series; as well as the mode of investigation used in Art. 5 *et seq.* only changing sometimes greater for less, or less for greater. Thus then, reasoning from the table of successive sums in Art. 3, in which s is alternately above and below the expressions $0, a, a - b, a - b + c, \&c.$ because 0 is below, and a above the value s of the series $a - b + c - d + \&c.$ but 0 nearer than a to that value, it follows that s lies between

tween 0 and $\frac{1}{2}a$, and that $\frac{1}{2}a$ is greater than s , but nearer to s than a is. In like manner, because a is above, and $a - b$ below the value s , but a nearer to that value than $a - b$ is, it follows that s lies between a and $a - b$, and that the arithmetical mean $a - \frac{1}{2}b$ is below s , but that it is nearer to s than $a - b$ is. And thus, the same reasoning holding in every pair of successive sums, the arithmetical means between them will form another series of terms, which are alternately greater and less than s the value of the proposed series; but here greater and less in the contrary way to what they were for the converging series, namely, those steps greater here which were less there, and less here which before were greater. And this first set of arithmetical means, will either converge to the value of s , or will at least diverge less from it than the progression of successive sums. Again, the same reasoning still holding good, by taking the arithmetical means of those first means, another set is found, which will either converge, or else diverge less than the former. And so on as far as we please, every new operation gradually checking the first or greatest divergency, till a number of the first terms of a set converge sufficiently fast, to afford a near value of s the proposed series.

10. Or, by taking the first terms of all the orders of means, we find the same set of theorems, namely

$$\frac{a}{2}, \frac{3a-b}{4}, \frac{7a-4b+c}{8}, \frac{15a-11b+5c-d}{16}, \text{ \&c. or in general, } \frac{1}{2^n} \times \{ 2^n - 1 \cdot a - A - n \cdot b + B - n \cdot \frac{n-1}{2} \cdot c - \text{ \&c. which$$

will be alternately above and below s the value of the series, till the divergency is overcome. Then this series, which consists of the first terms of the several orders of means, may be treated as the successive sums, taking several orders of means of these again. After which the first terms of these last orders may be treated again in the same manner; and so on as far as we please. Or the series of second terms, or third terms, &c. or sometimes, the terms ascending obliquely, may be treated in the same manner to advantage. And with

The construction and continuation of this table, is a business of little labour. For the numbers in the first horizontal line next below the line of the powers of 2, are those powers diminished each by unity. The numbers in the next horizontal line, are made from the numbers in the first, by subtracting from each the index of that power of 2 which stands above it. And for the rest of the table, the formation of it is obvious from this principle, which reigns through the whole, that every number in it is the sum of two others, namely of the next to it on the left in the same horizontal line, and the next above that in the same vertical column. So that the whole table is formed from a few of its initial numbers, by easy operations of addition.

In converging series, it will be farther useful, first to collect a few of the initial terms into one sum, and then apply our method to the following terms, which will be sooner valued because they will converge slower.

12. For the first example, let us take the very slowly converging series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \&c.$ which is known to express the hyp. log. of 2, which is $= .69314718$.

Here $a = 1$, $b = \frac{1}{2}$, $c = \frac{1}{3}$, $d = \frac{1}{4}$, $\&c.$ and the value, as found by theorem the 1st, 2d, 3d, 4th, 10th, and 20th, will be thus:

$$1st, \frac{a}{2} = \frac{1}{2} = .5.$$

$$2d, \frac{3a - b}{4} = \frac{3 - \frac{1}{2}}{4} = \frac{2\frac{1}{2}}{4} = .625.$$

$$3d, \frac{7a - 4b + c}{8} = \frac{7 - 2 + \frac{1}{3}}{8} = \frac{5\frac{1}{3}}{8} = \frac{2}{3} = .666666.$$

$$4th, \frac{15a - 11b + 5c - d}{16} = \frac{15 - 5\frac{1}{2} + 1\frac{2}{3} - \frac{1}{4}}{16} = .68229.$$

$$10th, \frac{1023a - 1013b + \&c.}{2^{10}} = \frac{709.698413}{4^5} = .693065.$$

$$20th, \frac{1048575a - 1048555b + \&c.}{2^{20}} = \frac{726817.45238043}{4^{10}} = .69314714.$$

Where it is evident that every theorem gives always a nearer value than the former: the 10th theorem gives the value true to the 4th figure,

figure, and the 20th theorem to the 8th figure. The operation for the 10th and 20th theorems, will be easily performed by dividing, mentally, the numbers in their respective columns in the table of coefficients in Art. 11, by the ordinate numbers 1, 2, 3, 4, 5, 6, &c. placing the quotients of the alternate terms below each other, then adding each up, and dividing the difference of the sums continually five or ten times successively by the number 4: after the manner as here placed below, where the operation is set down for both of them.

1. For the 10th Theorem.

+	-
1023	506.5
322.666667	212
127.6	64.333
25.142857	7
1.222222	0.1
1499.631746	789.933
789.933333	
4 709.698413	
4 177.424603	
4 44.356151	
4 11.089038	
4 2.772259	
	.693065

2. For the 20th Theorem.

+	-
1048575	524277.5
349455	261806.25
208476	171146.
141159.42857143	113824.5
87180.66666667	61666.6
39264.54545454	21995.83333333
10613.84615385	4318.57142857
1446.66666667	387.25
79.47058824	11.72222222
1.10526316	0.05
1886251.72936456	1159434.27698413
1159434.27698413	
4 726817.45238043	
4 181704.36309511	
4 45426.09077378	
4 11356.52269345	
4 2839.13067336	
4 709.78266834	
4 177.44566708	
4 44.36141677	
4 11.09035419	
4 2.77258855	
	.69314714

Again, to perform the operation by taking the successive sums, and the arithmetical means: let the terms $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \&c.$ be reduced to decimal numbers, by dividing the common numerator 1 by the denominators 2, 3, 4, &c. or rather by taking these out of the table at the end of my *Miscellanea Mathematica*, published in 1775, which contains a table of the square roots and reciprocals of all the numbers,

1, 2, 3, 4, 5, 6, &c. to 1000, and which is of great use in such calculations as these. Then the operation will stand thus :

The terms.	Suc. fums.						
+ 1	1						
- 0.5	0.5						
+ 333333	833333						
- 25	583333						
+ 2	783333						
- 166666	616666						
+ 142857	759524	688095	692560	693056	693131	693144	693147
- 125	634524	697024	693552	693205	693158	693150	
+ 111111	745635	690080	692858	693110	693142		
- 1	645635	695635	693362	693173			
+ 090909	736544	691090	692984				
- 083333	653211	694878					

The several orders of means.

Here, after collecting the first twelve terms, I begin at the bottom, and, ascending upwards, take a very few arithmetical means between the successive sums, placing them on the right of them: it being unnecessary to take the means of the whole, as any part of them will do the business, but the lower ones in a converging series best, because they are nearer the value sought, and approach sooner to it. I then take the means of the first means, and the means of these again, and so on, till the value is obtained as near as may be necessary. In this process we soon distinguish whereabouts the value lies, the limits or means, which are alternately above and below it, gradually contracting, and approaching towards each other. And when the means are reduced to a single one, and it is found necessary to get the value more exactly, I then go back to the columns of successive sums, and find another first mean, either next below or above those before found, and continue it through the 2d, 3d, &c. means, which makes now a duplicate in the last column of means, and the mean between them gives another single mean of the next order; and so on as far as we see it necessary. By such a gradual progress we use no more terms nor labour than is quite requisite for the degree of accuracy required.

Or, after having collected the sum of any number of terms, we may apply any of the formulæ to the following terms. So, having as above

found $\cdot 653211$ for the sum of the first 12 terms, and calling the next or 13th term $\cdot 076923 = a$, the 14th term $\cdot 0714285 = b$, the next, $\cdot 06666$ &c. $= c$, and so on: then the 2d theorem $\frac{3a-b}{4}$ gives $\cdot 039835$, which added to $\cdot 653211$ the sum of the first 12 terms, gives $\cdot 693046$, the value of the series true in three places of figures; and the 3d theorem $\frac{7a-4b+c}{8}$ gives $\cdot 039927$ for the following terms, and which added to $\cdot 653211$ the sum of the first 12 terms, gives $\cdot 693138$, the value of the series true in five places. And so on.

13. For a second example, let us take the slowly converging series $\frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \frac{7}{6} + \text{\&c.}$ which is $= \frac{1}{2} + \text{hyp. log. of } 2 = 1\cdot 19314718$. Then

Terms.	Suc. fums.	Arithmetical means.				
+ 2	2					
- 1·5	0·5					
+ 1·333333	1·833333					
- 1·25	0·583333					
+ 1·2	1·783333					
- 1·166666	0·616666					
+ 1·142857	1·759524	1·188095				
- 1·125	0·634524	1·197024	1·192560			
+ 1·111111	1·745635	1·190080	1·193552	1·193056		
- 1·1	0·645635	1·195635	1·192858	1·193205	131	
+ 1·090909	1·736544	1·191090	1·193362	1·193110	157	144
- 1·083333	0·653211	1·194878	1·192984	1·193173	142	150
						147

Here, after the 3d column of means, the first four figures $1\cdot 193$, which are common, are omitted, to save room and the trouble of writing them so often down; and in the last three columns, the process is repeated with the last three figures of each number; and the last of these 147, joined to the first four, give $1\cdot 193147$ for the value of the series proposed. And the same value is also obtained by the theorems used as in the former example.

14. For the third example let us take the converging series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{\&c.}$ which is $= \cdot 7853981$ &c. or $\frac{1}{4}$ of the circumference of the circle whose diameter is 1. Here $a = 1$, $b = \frac{1}{2}$, $c =$

$c = \frac{1}{3}$, &c. then turning the terms into decimals, and proceeding with the successive sums and means as before, we obtain the 5th mean true within a unit in the 6th place as here below :

Terms.	S. sums.					
+ 1	666667					
- 0.333333	866667					
+ 2	723810					
- 142857	834921					
+ 111111	744012					
- 090909	820935					
+ 76923	754268					
- 66667	813091					
+ 58823	760459					
- 52632	808078					
+ 47619						
		Arithmetical means.				
		782474	785037	785339	785387	785397
		787601	785641	785434	785407	
		783680	785227	785380		
		786775	785522			
		784269				

15. To find the value of the converging series

$$1 - \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \&c.$$

which occurs in the expression for determining the time of a body's descent down the arc of a circle:

The first terms of this series I find ready computed by Mr. Baron Maferes, pa. 219 Philof. Transf. 1777; these being arranged under one another, and the sums collected, &c. as before, give .834625 for the value of that series, being only 1 too little in the last figure.

Terms.	S. sums.					
+ 1 .	75					
- 0.25	890625					
+ 140625	792969					
- 97656	867737					
+ 74768	807175					
- 60562	858064					
+ 50889	814185					
- 43879	852750					
+ 38565	818351					
- 34399	849396					
+ 31045						
		Arithmetical means.				
		832620	834372	834584	834618	834625
		836124	834796	834652	834631	
		833468	834509	834610		
		835550	834711			
		833873				

16. To find the value of $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \&c.$ consisting of the reciprocals of the natural series of square numbers.

Terms.	S. fums.						
+ 1							
- 0.25	75						
+ 111111	861111						
- 625	798611						
+ 4	838611						
- 27778	810833						
+ 20408	831241						
- 15625	815616	823429	822609	822492	822472	822468	822467
+ 12346	827962	821789	822376	822452	822452	822464	822466
- 1	817962	822962	822528	822476	822468	822466	822467
+ 08264	826226	822094	822424	822460	822468	822466	822467
- 6944	819282	822754	822497	822460	822468	822466	822467
+ 5917	825199	822240	822497	822460	822468	822466	822467

Arithmetical means.

The last mean $\cdot 822467$ is true in the last figure, the more accurate value of the series $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \&c.$ being $\cdot 8224670 \&c.$

17. Let the diverging series $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \&c.$ be proposed; where the terms are the reciprocals of those in Art. 13.

Terms.	Suc. fums.						
+ 5	+ 5						
- 666666	- 166666						
+ 75	+ 583333						
- 8	- 216666						
+ 833333	+ 616666						
- 857143	- 240476	188095	192560	193056	131	144	147
+ 875	+ 634524	197024	193552	193205	157	150	147
- 888889	- 254365	190080	192858	193110	142	150	147
+ 9	+ 645635	195635	193362	193173	142	150	147
- 909091	- 263456	191090	193362	193173	142	150	147
+ 916667	+ 653211	194878	192984	193173	142	150	147

Arithmetical means.

Here the successive fums are alternately $+$ and $-$, as well as the terms themselves of the proposed series, but all the arithmetical means are positive. The numbers in each column of means are alternately too great and too little, but so as visibly to approach towards each other. The same mutual approximation is visible in all the oblique lines from left to right, so that there is a general and mutual tendency, in all the columns, and in all the lines, to the limit of the value of the series. But with this difference, that all the numbers in any line descending obliquely from left to right, are on one side of the limit, and those

those in the next line in the same direction, all on the other side, the one line having its numbers all too great, while those in the next line are all too little; but, on the contrary, the lines which ascend from below obliquely towards the right, have their numbers alternately too great and too little, after the manner of those in the columns, but approximating quicker than those in the columns. So that, after having continued the columns of arithmetical means to any convenient extent, we may then select the terms in the last, or any other line obliquely ascending from left to right, or rather beginning with the last found mean on the right, and descending towards the left; then arrange those terms below one another in a column, and take their continual arithmetical means, like as was done with the first successive sums, to such extent as the case may require. And if neither these new columns, nor the oblique lines approach near enough to each other, a new set may be formed from one of their oblique lines which has its terms alternately too great and too little. And thus we may proceed as far as we please. These repetitions will be more necessary in treating series which diverge more; and having here once for all described the properties attending the series, with the method of repetition, we shall only have to refer to them as occasion shall offer. In the present instance, the last two or three means vary or differ so little, that the limit may be concluded to lie nearly in the middle between them, and therefore the mean between the two last 144 and 150, namely 147, may be concluded to be very near the truth, in the last three figures; for as to the first three figures 193, I dropt the repetition of them after the first three columns of means, both to save space, and the trouble of writing them so often over again. So that the value of the series in question may be concluded to be $\cdot 193147$ very nearly, which is $= -\frac{1}{2} +$ the hyp. log. of 2; or 1 less than its reciprocal series in Art. 13.

18. Take the diverging series $\frac{5}{4} - \frac{5 \cdot 7}{4 \cdot 6} + \frac{5 \cdot 7 \cdot 9}{4 \cdot 6 \cdot 8} - \frac{5 \cdot 7 \cdot 9 \cdot 11}{4 \cdot 6 \cdot 8 \cdot 10} + \&c.$

Here, first using some of the formulæ, we have by the

1st,

$$1^{\text{st}}, \frac{a}{2} = .625$$

$$2^{\text{d}}, \frac{3a - b}{4} = .57292$$

$$3^{\text{d}}, \frac{7a - 4b + c}{8} = .56966$$

$$4^{\text{th}}, \frac{15a - 11b + 5c - d}{16} = .56917$$

$$5^{\text{th}}, \frac{31a - 26b + 16c - 6d + e}{32} = .56907. \text{ \&c.}$$

Or, thus, taking the several orders of means, &c.

Terms.	Suc. fums.	Arithmetical means.				
+ 1.25	+ 1.25	520833	566406	8685	8970	021
- 1.458333	- 0.208333	611980	570964	9255	9072	043
+ 1.640625	+ 1.432292	529948	567546	8889	9015	032
- 1.804688	- 0.372396	605144	570232	9141	9050	038
+ 1.955079	+ 1.582683	535320	568050	8959		035
- 2.094727	- 0.512044	600780	569868			
+ 2.225647	+ 1.713603	538956				
- 2.349294	- 0.635691					

Here the successive fums are alternately + and -, but the arithmetical means are all +. After the second column of means, the first two figures 56 are omitted, being common; and in the last three columns the first three figures 569, which are common, are omitted. Towards the end, all the numbers, both oblique and vertical, approach so near together, that we may conclude that the last three figures 035 are all true; and these being joined to the first three 569, we have .569035 for the value of the series, which is otherwise found $= \frac{2+\sqrt{2}}{6} = .56903559 \text{ \&c.}$

19. Let us take the diverging series

$$\frac{2^2}{1} - \frac{3^2}{2} + \frac{4^2}{3} - \frac{5^2}{4} + \text{\&c.} \text{ or } \frac{4}{1} - \frac{9}{2} + \frac{16}{3} - \frac{25}{4} + \text{\&c.}$$

$$\text{or } 4 - 4\frac{1}{2} + 5\frac{1}{3} - 6\frac{1}{4} + 7\frac{1}{5} - 8\frac{1}{6} + \text{\&c.}$$

Terms.

Terms.	Sums.	Arithmetical means.			
+ 4	+ 4				
- 4.5	- 0.5				
+ 5.333333	+ 4.833333				
- 6.25	- 1.416666				
+ 7.2	+ 5.783333				
- 8.166666	- 2.383333	2.188096	1.942560	059	128
+ 9.142857	+ 6.759524	1.697024	1.943557	207	158
- 10.125	- 3.365476	2.190080	1.942857	110	142
+ 11.111111	+ 7.745635	1.695635	1.943362	173	143
- 12.1	- 4.354365	2.191089	1.942983		150
+ 13.090909	+ 8.736544	1.694877			147
- 14.083333	- 5.346789				

After the second column of means, the first four figures 1.943 are omitted, being common to all the following columns; to these annexing the last three figures 147 of the last mean, we have 1.943147 for the sum of the series, which we otherwise know is equal to $\frac{1}{4} + \text{hyp. log. of } 2$. See Simp. Differt. Ex. 2. p. 75 and 76.

And the same value might be obtained by means of the formulæ, using them as before.

20. Taking the diverging series $1 - 2 + 3 - 4 + 5 - \&c.$ the method of means gives us,

Terms.	Sums.	Means.
+ 1	+ 1	0
- 2	- 1	$\frac{1}{2}$
+ 3	+ 2	0
- 4	- 2	$\frac{1}{2}$
+ 5	+ 3	0
- 6	- 3	$\frac{1}{2}$

Where the second, and every succeeding column of means, gives $\frac{1}{4}$ for the value of the series proposed.

In like manner, using the theorems, the first gives $\frac{1}{2}$, but the second, third, fourth, &c. give each of them the same value $\frac{1}{4}$; thus:

$$\begin{aligned} \frac{a}{2} &= \frac{1}{2} \\ \frac{3a - b}{4} &= \frac{3 - 2}{4} = \frac{1}{4} \\ \frac{7a - 4b + c}{8} &= \frac{7 - 8 + 3}{8} = \frac{2}{8} = \frac{1}{4} \\ \frac{15a - 11b + 5c - d}{16} &= \frac{15 - 22 + 15 - 4}{16} = \frac{4}{16} = \frac{1}{4} \cdot \&c. \end{aligned}$$

21. Taking the series $1 - 4 + 9 - 16 + 25 - 36 + \&c.$ whose terms consist of the squares of the natural series of numbers, we have, by the arithmetical means,

Terms.	Sums.	Arithmetical means.		
+ 1	+ 1	- 1	+ $\frac{1}{4}$	0
- 4	- 3	+ $1\frac{1}{2}$	- $\frac{1}{4}$	0
+ 9	+ 6	- 2	+ $\frac{1}{4}$	0
- 16	- 10	+ $2\frac{1}{2}$	- $\frac{1}{4}$	0
+ 25	+ 15	- 3		
- 36	- 21			

Where it is only in the second column of means that the divergency is counteracted; after that the third and all the other orders of means give 0 for the value of the series $1 - 4 + 9 - 16 + \&c.$

The same thing takes place on using the formulæ, for

$$\frac{a}{2} = \frac{1}{2}$$

$$\frac{3a - b}{4} = \frac{3 - 4}{4} = -\frac{1}{4}$$

$$\frac{7a - 4b + c}{8} = \frac{7 - 16 + 9}{8} = \frac{0}{8} = 0$$

$$\frac{15a - 11b + 5c - d}{16} = \frac{15 - 44 + 45 - 16}{16} = \frac{0}{16} = 0$$

where the third and all after it give the same value 0.

22. Taking the geometrical series of terms $1 - 2 + 4 - 8 + \&c.$ then

Terms.	Sums.	Arithmetical Means.			
+ 1	+ 1	+ $\frac{1}{2}$	+ $\frac{1}{4}$	$\frac{3}{8}$	$\frac{5}{16}$
- 2	- 1	+ 1	+ $\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{8}$
+ 4	+ 3	- 1	0	$\frac{1}{2}$	$\frac{1}{4}$
- 8	- 5	+ 3	+ 1	0	$\frac{1}{2}$
+ 16	+ 11	- 5	- 1	+ 1	0
- 32	- 21	+ 11	+ 3	- 1	+ 1
+ 64	+ 43	- 21	- 5	+ 3	- 1
- 128	- 85	+ 43	+ 11	- 5	$\&c.$
+ 256	+ 171				

Here

Here the lower parts of all the columns of means, from the cipher 0 downwards, consist of the same series of terms $+ 1 - 1 + 3 - 5 + 11 - 21 + 43 - 85 + \&c.$ and the other part of the columns, from the cipher upwards, as well as each line of oblique means, parallel to, and above the line of ciphers, forms a series of terms $\frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{5}{16}, \dots, \frac{1}{3} \cdot \frac{2^n + 1}{2^n}$, alternately above and below the value of the series, $\frac{1}{3}$, and approaching continually nearer and nearer to it, and which, when infinitely continued, or when n is infinite, the term becomes $\frac{1}{3}$ for the value of the geometrical series, $1 - 2 + 4 - 8 + 16 - \&c.$

And the same set of terms would be given by each of the formulæ.

23. Take the geometrical series $1 - 3 + 9 - 27 + 81 - \&c.$ Then

Terms.	Sums.	Arithmetical means.									
$+ 1$	$+ 1$	$+$	$\frac{1}{2}$	0	$+$	$\frac{1}{2}$	0	$+$	$\frac{1}{2}$	0	$+$
$- 3$	$- 2$	$+$	$2\frac{1}{2}$	$+$	1	$+$	$\frac{1}{2}$	$+$	$\frac{1}{2}$	0	$+$
$+ 9$	$+ 7$	$-$	$6\frac{1}{2}$	$-$	2	$+$	$2\frac{1}{2}$	$+$	1	$+$	$\frac{1}{2}$
$- 27$	$- 20$	$+$	$20\frac{1}{2}$	$+$	7	$-$	$6\frac{1}{2}$	$-$	2	$+$	$2\frac{1}{2}$
$+ 81$	$+ 61$	$-$	20	$-$	20	$+$	7	$+$	$2\frac{1}{2}$	$+$	1

Here the column of successive sums, and every second column of the arithmetical means, below the 0, consists of the same series of terms $1, - 2, + 7, - 20, + \&c.$ whilst all the other columns of means consist of this other set of terms $\frac{1}{2}, - \frac{1}{2}, + 2\frac{1}{2}, - 6\frac{1}{2}, + \&c.$ also the first oblique line of means, $\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \&c.$ consists of the terms $\frac{1}{2}$ and 0 alternately, which are all at equal distance from the value of the series proposed $1 - 3 + 9 - 27 + 81 - \&c.$ as indeed are the terms of all the other oblique descending lines. And the mean between every two terms gives $\frac{1}{3}$ for that value. And the same terms would be given by the formulæ, namely alternately $\frac{1}{2}$ and 0.

And thus the value of any geometrical series, whose ratio or second term is r , will be found to be $= \frac{1}{1+r}.$

24. Finally, let there be taken the hypergeometrical series $1 - 1 + 2 - 6 + 24 - 120 + \&c. = 1 - 1A + 2B - 3C + 4D - 5E + \&c.$ which difficult series has been honoured by a very considerable memoir written upon the valuation of it by the late famous L. Euler, in the New Petersburg Commentaries, vol. v. where the value of it is at length determined to be $\cdot 5963473 \&c.$

To simplify this series, let us omit the first two terms $1 - 1 = 0$, which will not alter the value, and divide the remaining terms by 2, and the quotients will give $1 - 3 + 12 - 60 + 360 - 2520 + \&c.$ which, being half the proposed series, ought to have for its value the half of $\cdot 596347 \&c.$ namely $\cdot 298174$ nearly.

Now, ranging the terms in a column, and taking the sums and means as usual, we have

Terms.	Sums.	Arithmetical means.									
+	1	+	1	+	$\frac{1}{2}$	+	0	+	$\frac{3}{8}$	+	$1 \cdot 1250$
-	3	-	2	-	$\frac{3}{2}$	-	$1 \frac{3}{4}$	-	$3 \frac{1}{8}$	-	$10 \cdot 1875$
+	12	+	10	+	4	+	8	+	$23 \frac{1}{2}$	+	$4 \cdot 53125$
-	60	-	50	-	20	-	55	-	$177 \frac{1}{2}$	-	$33 \cdot 40625$
+	360	+	310	+	130	+	410	+	1525	+	$673 \cdot 75$
-	2520	-	2210	-	950	-	410	-	1525	-	$298 \cdot 375$
+	20160	+	17950	+	7870	+	3460	+	1525	+	$132 \cdot 484375$
											$59 \cdot 023438$

Where it is evident, that the diverging is somewhat diminished, but not quite counteracted, in the columns and oblique descending lines from beginning to end, as the terms in those directions still increase, though not quite so fast as the original series; and that the signs of the same terms are alternately $+$ and $-$, while those of the terms in the other lines obliquely ascending from left to right, are alternately one line all $+$, and another line all $-$, and these terms continually decreasing. The terms in the oblique descending lines, being alternately too great and too little, are the fittest to proceed with again. Take therefore any one of those lines, as suppose the first, and ranging it vertically, take the means as before, and they will approach nearer to the value of the series, thus:

+	·5	+	·25	+	·34375	+	·25	+	·361328	+	·194336	+	·492066
-	·0	+	·4375	+	·15625	+	·472656	+	·027344	+	·789795		
+	·875	-	·1250	+	·789062	-	·417969	+	·552246				
-	1·125	+	1·703125	-	1·625	+	3·522461						
+	4·53125	-	4·953125	+	8·669922								
-	14·4375	+	22·292969										
+	59·023438												

Here the same approximation in the lines and columns, towards the value of the series, is observable again, only in a higher degree; also the terms in the columns and oblique descending lines, are again alternately too great and too little, but now within narrower limits, and the signs of the terms are more of them positive; also the terms in each oblique ascending line, are still either all above or all below the value of the series, and that alternately one line after another as before. But the descending lines will again be the fittest to use, because the terms in each are alternately above and below the value sought. Taking therefore again the first of these oblique descending lines, treat it as before, and we shall obtain sets of terms approaching still nearer to the value, thus:

25	296875	296875	299073	297791	298306
34375	296878	301271	296509	298821	
25	305664	291748	301132		
361328	277832	310516			
194336	343201				
492066					

Here the approach to an equality, among all the lines and columns, is still more visible, and the deviations restricted within narrower limits, the terms in the oblique ascending lines still on one side of the value, and gradually increasing, while the columns and the oblique descending lines, for the most part, have their terms alternately too great and too little, as is evident from their alternately becoming greater and less than each other: and from an inspection of the whole, it is easy to pronounce that the first three figures of the number sought, will be 298. Taking therefore the last four terms of the first descending line, and proceeding as before, we have

$$\begin{array}{c|c|c|c}
 296875 & 297974 & 298203 & 298222 \\
 299073 & 298432 & 298240 & \\
 297791 & 298048 & & \\
 298306 & & &
 \end{array}$$

And, finally, taking the lowest ascending line, because it has most the appearance of being alternately too great and too little, proceed with it as before, thus :

$$\begin{array}{c|c|c|c}
 298306 & 298177 & 298161 & 298174 \\
 298048 & 298144 & 298187 & \\
 298240 & 298231 & & \\
 298222 & & &
 \end{array}$$

where the numbers in the lines and columns gradually approach nearer together, till the last mean is true to the nearest unit in the last figure, giving us $\cdot 298174$ for the value of the proposed hypergeometrical series $1 - 3 + 12 - 60 + 360 - 2520 + 20160 - \&c.$

And in like manner are we to proceed with any other series whose terms have alternate signs.

Royal Military Acad.
Woolwich, May, 1780.

P O S T S C R I P T.

SINCE the foregoing method was discovered, and made known to several friends, two passages have been offered to my consideration, which I shall here mention, in justice to their authors, Sir Isaac Newton, and the late learned Mr. Euler.

The first of these is in Sir Isaac's letter to Mr. Oldenburg, dated *October* 24, 1676, and may be seen in Collins's *Commercium Epistolicum*, p. 177, the last paragraph near the bottom of the page, namely, *Per seriem*

seriem Leibnitii etiam, si ultimo loco dimidium termini adjiciatur, & alia quædam similia artificia adhibeantur, potest computum produci ad multas figuras. The series here alluded to, is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} +$ &c. denoting the area of the circle whose diameter is 1; and Sir Isaac here directs to add in half the last term, after having collected all the foregoing, as the means of obtaining the sum a little exacter. And this, indeed, is equivalent to taking one arithmetical mean between two successive sums, but it does not reach the idea contained in my method. It appears also, both by the other words, *& alia quædam similia artificia adhibeantur*, contained in the above extract, and by these, *alias artes adhibuisse*, a little higher up in the same page 177, that Sir Isaac Newton had several other contrivances for obtaining the sums of slowly converging series; but what they were, it may perhaps be now impossible to determine.

The other is a passage in the *Novi Comment. Petropol.* tom. v. p. 226, where Mr. Euler gives an instance of taking one set of arithmetical means between a series of quantities which are gradually too little and too great, to obtain a nearer value of the sum of a series in question. But neither does this reach the idea contained in my method. However, I have thought it but justice to the characters of these two eminent men, to make this mention of their ideas, which have some relation to my own, though unknown to me at the time of my discovery.



III.

A Method of summing the Series $a + bx + cx^2 + dx^3 + ex^4 + \&c.$ when it converges very slowly, namely, when x is nearly equal to 1, and the Coefficients $a, b, c, d, \&c.$ decrease very slowly: the Signs of all the Terms being positive.

Art. 1. **W**HEN we have occasion to find the sum of such series as that above-mentioned, having the coefficients $a, b, c, d, \&c.$ of the terms, decreasing very slowly, and the converging quantity x pretty large; we can neither find the sum by collecting the terms together, on account of the immense number of them which it would be necessary to collect; neither can it be summed by means of the differential series, because the powers of the quantity $\frac{x}{1-x}$ will then diverge faster than the differential coefficients converge. In such case then we must have recourse to some other method of transforming it into another series which shall converge faster. The following is a method by which the proposed series is changed into another, which converges so much the quicker as the original series is slower.

2. The method is thus. Assume $\frac{a^2}{D} =$ the given series

$a + bx + cx^2 + dx^3 + \&c.$ Then shall

$D = \frac{a^2}{a + bx + cx^2 + \&c.}$; which, by actual division, is =

$$a - \frac{bx}{a} - c - \frac{b^2}{a} \cdot x^2 - d - \frac{2bc}{a} + \frac{b^3}{a^2} \cdot x^3 - e - \frac{2bd + c^2}{a} + \frac{3b^2c}{a^2} - \frac{b^4}{a^3} \cdot x^4 - \&c.$$

Consequently a^2 divided by this series will be equal to the series proposed, and this new series will be very easily summed, in comparison with the original one, because all the coefficients after the second term are evidently very small; and indeed they are so much the smaller, and fitter for summation, by how much the coefficients of the original

original series are nearer to equality ; so that when these $a, b, c, d,$ &c. are quite equal, then the third, fourth, &c. coefficients of the new series become equal to nothing, and the sum accurately equal to

$\frac{a^2}{a-bx} = \frac{a^2}{a-ax} = \frac{a}{1-x}$; which we also know to be true from other principles.

3. Although the first two terms, $a - bx$, of the new series be very great in comparison with each of the following terms, yet these latter may not always be small enough to be entirely rejected where much accuracy is required in the summation. And in such case it will be necessary to collect a great number of them, to obtain their sum pretty near the truth ; because their rate of converging is but small, it being indeed pretty much like to the rate of the original series, but only the terms, each to each, are much smaller, and that commonly in a degree to the hundredth or thousandth part.

4. The resemblance of this new series however, beginning with the third term, to the original one, in the law of progression, intimates to us that it will be best to sum it in the very same manner as the former. Hence then putting

$$\begin{aligned} a' &= c - \frac{b^2}{a} \\ b' &= d - \frac{2bc}{a} + \frac{b^3}{a^2} \\ c' &= e - \frac{2bd + c^2}{a} + \frac{3b^2c}{a^2} - \frac{b^4}{a^3} \\ &\text{\&c.} \end{aligned}$$

and consequently the proposed series $a + bx + cx^2 + \text{\&c.}$

$$= \frac{a^2}{a - bx - a'x^2 - b'x^3 - c'x^4 - \text{\&c.}} = \frac{a^2}{a - bx - x^2 \times (a' + b'x + c'x^2 + \text{\&c.})}$$

by taking the sum of the series $a' + b'x + c'x^2 + \text{\&c.}$ by the very
2 fame

same theorem as before, the sum S of the original series will then be expressed thus,

$$S = \frac{a^2}{a - bx - \frac{a'^2 x^2}{a' - b'x - c' - \frac{b'^2}{a'} \cdot x^2 - d' - \frac{2b'c'}{a'} + \frac{b'^3}{a'^2} \cdot x^3 - \&c.}}$$

where the series in the last denominator, having again the same properties with the former one, will have its first two terms very large in respect of the following terms. But these first two terms, $a' - b'x$, are themselves very small in comparison with the first two, $a - bx$, of the former series; and therefore much more are the third, fourth, &c. terms of this last denominator very small in comparison with the same $a - bx$: for which reason they may now perhaps be small enough to be neglected.

5. But if these last terms be still thought too large to be omitted, then find the sum of them by the very same theorem: and thus proceed, by repeating the operation in the same manner, till the required degree of accuracy is obtained. Which it is evident, will happen after a small number of repetitions, because that, in each new denominator, the third, fourth, &c. terms are commonly depressed, in the scale of numbers, two or three places lower than the first and second terms are. And the general theorem, denoting the sum S when the process is continually repeated, will be this,

$$S = \frac{a^2}{a - bx - \frac{a'^2 x^2}{a' - b'x - \frac{a''^2 x^2}{a'' - b''x - \frac{a'''^2 x^2}{a''' - b'''x - \frac{a^{(4)} x^2}{a^{(4)} - b^{(4)}x - \&c.}}}}}}$$

6. But

6. But the general denominator D in the fraction $\frac{a^2}{D}$, which is assumed for the value of S or $a + bx + cx^2 + \&c.$ may be otherwise found as below; from which the general law of the values of the coefficients will be rendered visible. Assume S or $a + bx + cx^2 + \&c.$

$$\text{or } \frac{a^2}{D} = \frac{a^2}{a - bx - a'x^2 - b'x^3 - c'x^4 - \&c.}; \text{ then shall}$$

$$\begin{aligned} a^2 &= a + bx + cx^2 + \&c. \times a - bx - a'x^2 - b'x^3 - \&c. \\ &= a^2 + abx + acx^2 + adx^3 + aex^4 + afx^5 + \&c. \\ &\quad - ab \quad - bb \quad - bc \quad - bd \quad - be \\ &\quad \quad - a'a \quad - a'b \quad - a'c \quad - a'd \\ &\quad \quad \quad - b'a \quad - b'b \quad - b'c \\ &\quad \quad \quad \quad - c'a \quad - c'b \\ &\quad \quad \quad \quad \quad - d'a \end{aligned}$$

Hence, by equating the coefficients of the like terms to nothing, we obtain the following general values:

$$\begin{aligned} a' &= c - \frac{bb}{a} \\ b' &= d - \frac{ba' + cb}{a} \\ c' &= e - \frac{bb' + ca' + db}{a} \\ d' &= f - \frac{bc' + cb' + da' + eb}{a} \\ e' &= g - \frac{bd' + cc' + db' + ea' + fb}{a} \\ &\&c. \end{aligned}$$

Where the values of the coefficients are the same in effect as before found, but here the law of their continuation is manifest

7. To exemplify now the use of this method, let it be proposed to sum the very flow series

$$x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6 + \&c.$$

when $x = \frac{9}{10} = .9$, which denotes the hyperbolic log. of $\frac{1}{1-x}$, or in this case of 10.

Now it will be proper, in the first place, to collect a few of the first terms together, and then apply the theorem to the remaining terms, which, by being nearer to an equality than the terms are near the beginning of the series, will be fitter to receive the application of the theorem. Thus to collect the first 12 terms:

No.	Powers of x	The first 12 terms, found by dividing $x, x^2, x^3, \&c.$ by the numbers 1, 2, 3, &c.
1	.9 - - - -	.9
2	.81 - - - -	.405
3	.729 - - - -	.243
4	.6561 - - - -	.164025
5	.59049 - - - -	.118098
6	.531441 - - - -	.0885735
7	.4782969 - - - -	.06832812857
8	.43046721 - - - -	.05380840125
9	.387420489 - - - -	.043046721
10	.3486784401 - - - -	.03486784401
11	.31381059609 - - - -	.02852823601
12	.282429536481 - - - -	.02353579471
13	.2541865828329	2.17081162555 the sum of 12 terms.

Then we have to find the sum of the rest of the terms after these first 12, namely of $x^{13} \times : \frac{1}{13} + \frac{1}{14}x + \frac{1}{15}x^2 + \frac{1}{16}x^3 + \&c.$ in which $x = .9$, and $x^{13} = .2541865828329$; also $a = \frac{1}{13}$, $b = \frac{1}{14}$, $c = \frac{1}{15}$, &c. and the first application of our rule, gives, for the value of $\frac{1}{13} + \frac{1}{14}x + \frac{1}{15}x^2 + \&c.$ or S,

$$\frac{1}{13} = .005917159763 \&c.$$

$$.012637363 - x^2 \times : .000340136 + .000279397x + .000233592x^2 + .000198295x^3 + .000170585x^4 + .000148459x^5 + \&c.$$

the second gives

$$\begin{array}{r} .00591715976 \\ \hline .000340136^2 x^2 \\ \hline .012637363 - \end{array}$$

$$.000088678 - x^2 \times : .000004086 + .000003060x + .000002380x^2 + .000001898x^3$$

the third gives

$$\begin{array}{r} .00591715976 \\ \hline .000340136^2 x^2 \\ \hline .012637363 - \end{array}$$

$$\begin{array}{r} .000088678 - \\ \hline .000004087^2 x^2 \\ \hline .000001333 - x^2 \times : .000000089 + .000000050x \end{array}$$

the

the fourth gives

$$\begin{array}{r}
 \cdot 00591715976 \\
 \hline
 \cdot 012637363 - \frac{\cdot 000340136^2 x^2}{000004087^2 x^2} \\
 \cdot 000088678 - \frac{\cdot 000000089^2 x^2}{\cdot 0000000344} \\
 \cdot 000001333 - \frac{\cdot 0000000344}{\cdot 0000000344}
 \end{array}$$

Or, when the terms in the numerators are squared, it is

$$\begin{array}{r}
 \cdot 00591715976 \\
 \hline
 \cdot 012637363 - \frac{\cdot 000000093710985}{\cdot 000000000013526212} \\
 \cdot 000088678 - \frac{\cdot 000000000000006416}{\cdot 0000000344} \\
 \cdot 000001333 - \frac{\cdot 0000000344}{\cdot 0000000344}
 \end{array}$$

Or, by omitting a proper number of ciphers, it is

$$\begin{array}{r}
 \cdot 0591715976 \\
 \hline
 \cdot 12637363 - \frac{\cdot 0093710985}{\cdot 013526212} \\
 \cdot 88678 - \frac{\cdot 006416}{\cdot 1333 - \frac{\cdot 344 - z}{\cdot 344 - z}}
 \end{array}$$

I have written an unknown quantity z after the last denominator, to represent the small quantity to be subtracted from the last denominator 344. Now, rejecting the small quantity z , and beginning at the last fraction to calculate, their values will be as here ranged in the first annexed column.

Fractions.	I. Ra.	II. Ra.	III. Ra.	IV. Ratio.
$\cdot 518200000$	425	4.01	2.39	
1218931	106	1.68	$\frac{2.39 \times 63z}{1.68 \times 187}$	$\frac{2.39 \times 63z}{1.68 \times 187}$
11799	63	$\frac{63z}{187}$	$\frac{1.68 \times 187}{63z}$	
187	$\frac{187}{z}$	187		
z	z			
$4\frac{3}{10}$	44	1.43	1.18	2.03

placing

placing z below them for the next unknown fraction. Divide then every fraction by the next below it, placing the quotients or ratios in the next column. Then take the quotients or ratios of these; and so on till the last ratio $\frac{2.39 \times 63z}{1.68 \times 187}$; which, from the nature of the series of the first terms of every column, must be less than the next preceding one 2.39: consequently z must be less than $\frac{1.68 \times 187}{63}$, or less than 5. But, from the nature of the series in the vertical row or column of first ratios, $\frac{187}{z}$ must be less than 63; and consequently z must be greater than $\frac{187}{63}$, or greater than 3. Since then z is less than 5 and greater than 3, it is probable that the mean value 4 is near the truth: and accordingly taking 4 for z , or rather 4.3, as z appears to be nearer 5 than 3, and taking the continual ratios, as placed along the last line of the table, their values are found to accord very well with the next preceding numbers, both in the columns and oblique rows.

Hence, using 0.43 for z in the denominator .344 — z of the last fraction of the general expression, and computing from the bottom, upwards through the whole, the quotients, or values of the fractions, in the inverted order, will be

$$\begin{array}{r} 213 \\ 12079 \\ 1223397 \\ .518414000 \end{array}$$

of which the last must be nearly the value of the series $\frac{1}{1.3} + \frac{1}{1.4}x + \frac{1}{1.5}x^2 + \&c.$ when $x = .9$.

Then this value .518414 of the series, being multiplied by x^{13} or .2541865828329, gives .1317738 for the sum of all the terms of the original series after the first 12 terms, to which therefore the sum of the first 12 terms, or 2.17081162, being added, we have 2.30258542 for the sum of the original series $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \&c.$ Which value is true within about 3 in the 8th place of figures, the more accurate value being 2.30258509 &c. or the hyp. log. of 10.



*The Investigation of certain easy and General Rules, for Extracting any
Root of a given Number.*

1. **T**HE roots of given numbers are commonly to be found, with much ease and expedition, by means of logarithms, when the indices of such roots are simple numbers, and the roots are not required to a great number of figures. And the square or cubic roots of numbers, to a good practical degree of accuracy, may be obtained, almost by inspection, by means of my tables of squares and cubes, published by order of the Commissioners of Longitude, in the year 1781. But when the indices of such roots are certain complex or irrational numbers, or when the roots are required to be found to a great many places of figures, it is necessary to make use of certain approximating rules, by means of the ordinary arithmetical computations. Such rules as are here alluded to, have only been discovered since the great improvements in the modern algebra: and the persons who have best succeeded in their enquiries after such rules, have been successively Sir Isaac Newton, Mr. Raphson, M. de Lagny, and Dr. Halley; who have shewn that the investigation of such theorems is also useful in discovering rules for approximating to the roots of all sorts of affected algebraical equations, to which the former rules, for the roots of all simple equations, bear a considerable affinity. It is presumed that the following short tract contains some advantages over any other method that has hitherto been given, both as to the ease and universality of the conclusions, and the general way in which the investigations are made: for here, a theorem is discovered, which, although it be general for all roots whatever, is at the same time very accurate, and so simple and

easy to use and to keep in mind, that nothing more so can be desired or hoped for; and farther, that instead of searching out rules severally for each root, one after another, our investigation is at once for any indefinite possible root, by whatever quantity the index is expressed, whether fractional, or irrational, or simple, or compound.

2. In every theorem, or rule, here investigated, let N be the given number, whose root is sought, n the index of that root, a its nearest rational root, or a^n the nearest rational power to N , whether greater or less, x the remaining part of the root sought, which may be either positive or negative, namely, positive when N is greater than a^n , otherwise negative.

Hence then the given number N is $= \overline{a + x}^n$, and the required root $N^{\frac{1}{n}} = a + x$.

3. Now, for the first rule, expand the quantity $\overline{a + x}^n$ by the binomial theorem, so shall we have

$$N = \overline{a + x}^n = a^n + n a^{n-1} x + n \cdot \frac{n-1}{2} a^{n-2} x^2 + \&c.$$

Subtract a^n from both sides, so shall

$$N - a^n = n a^{n-1} x + n \cdot \frac{n-1}{2} a^{n-2} x^2 + \&c.$$

Divide by $n a^{n-1}$, so shall

$$\frac{N - a^n}{n a^{n-1}} \text{ or } \frac{N - a^n}{n a^n} \times a = x + \frac{n-1}{2} \cdot \frac{x^2}{a} + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{x^3}{a^2} + \&c.$$

Here, on account of the smallness of the quantity x in respect of a , all the terms of this series, after the first term, will be very small, and may therefore be neglected without much error, which gives us $\frac{N - a^n}{n a^n} a$ for a near value of x , being only a small matter too great. And consequently $a + x = \frac{N + \frac{n-1}{2} a^n}{n a^n} a$ is nearly $= N^{\frac{1}{n}}$ the root sought. And this may be accounted the first theorem.

4. Again,

4. Again, let the equation $N = a^n + n a^{n-1} x + \&c.$ be multiplied by $n - 1$, and a^n added to each side, so shall we have

$$\overline{n - 1} \cdot N + a^n = n a^n + n - 1 \cdot n a^{n-1} x + \&c. \text{ for a divisor:}$$

Also multiply the sides of the same equation by a and subtract a^{n+1} from each, so shall we have

$$\overline{N - a^n} \cdot a = n a^n x + n \cdot \frac{n-1}{2} a^{n-1} x^2 + \&c. \text{ for a dividend:}$$

Divide now this dividend by the divisor, so shall

$$\frac{N - a^n}{n - 1 \cdot N + a^n} a = x - \frac{n-1}{2} \cdot \frac{x^2}{a} + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{x^3}{a} + \&c.$$

Which will be nearly equal to x , for the same reason as before; and this expression is nearly as much too little as the former expression was too great.

Consequently, by adding a , we have $a + x$ or $N^{\frac{1}{n}}$ nearly $= \frac{n N a}{n - 1 N + a^n}$ for a second theorem, and which is nearly as much in defect as the former was in excess.

5. Now because the two foregoing theorems differ from the truth by nearly equal small quantities, if we add together the two numerators and the two denominators of the foregoing two fractional expressions, namely $\frac{N + n - 1 \cdot a^n}{n a^n} a$ and $\frac{n N}{n - 1 \cdot N + a^n} a$, the sums will be the numerator and denominator of a new fraction, which will be much nearer than either of the former. The fraction so found is $\frac{n + 1 \cdot N + n - 1 \cdot a^n}{n - 1 \cdot N + n + 1 \cdot a^n} a$; which will be very nearly equal to $N^{\frac{1}{n}}$ or $a + x$ the root sought; for, by division, it is found to be equal to $a + x * - \frac{n-1}{2} \cdot \frac{n+1}{6} \cdot \frac{x^3}{a^2} + \&c.$ where the term is wanting which contains the square of x , and the following terms are very small. And this is the third theorem.

6. A fourth theorem might be found by taking the arithmetical mean
between

between the first and second, which would be $\frac{N + n-1 \cdot a^n}{n a^n} + \frac{n N}{n-1 \cdot N + a^n}$
 $\times \frac{a}{2}$; which will be nearly of the same value, though not so simple,
 as the third theorem; for this arithmetical mean is found equal to
 $a + x + \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{x^3}{a^2} + \&c.$

7. But the third theorem may be investigated in a more general way, thus:
 Assume a quantity of this form $\frac{pN + qa^n}{qN + pa^n} a$, with coefficients p and
 q to be determined from the process; the other letters N, a, n , repre-
 senting the same things as before; then divide the numerator by the
 denominator, and make the quotient equal to $a + x$, so shall the
 comparison of the coefficients determine the relation between p and q
 required. Thus,

$$pN + qa^n = \overline{p+q} \cdot a^n + pna^{n-1}x + pn \cdot \frac{n-1}{2} a^{n-2}x^2 + \&c.$$

$$qN + pa^n = \overline{p+q} \cdot a^n + qna^{n-1}x + qn \cdot \frac{n-1}{2} a^{n-2}x^2 + \&c.$$

then dividing the former of these by the latter, we have

$$\frac{pN + qa^n}{qN + pa^n} a \text{ or } a + x = a + \frac{p-q}{p+q} nx + \frac{p-q}{p+q} n \cdot \frac{n-1}{2} - \frac{qn}{p+q} \cdot \frac{x^2}{a} + \&c.$$

Then, by equating the corresponding terms, we obtain these three
 equations

$$a = a,$$

$$\frac{p-q}{p+q} n = 1,$$

$$\frac{n-1}{2} - \frac{qn}{p+q} = 0.$$

From which we find $\frac{p-q}{p+q} = \frac{1}{n}$ and $p : q :: n + 1 : n - 1$.

So that by substituting $n + 1$ and $n - 1$, or any quantities proportional
 to them, for p and q , we shall have $\frac{n+1 \cdot N + n-1 \cdot a^n}{n-1 \cdot N + n+1 \cdot a^n} a$ for the va-
 lue of the assumed quantity $\frac{pN + qa^n}{qN + pa^n} a$, which is supposed nearly
 equal to $a + x$, the required root of the quantity N .

8. Now

8. Now this third theorem $\frac{n+1.N+n-1.a^n}{n-1.N+n+1.a^n} a = N^{\frac{1}{n}}$, which is general for roots, whatever be the value of n , and whether a^n be greater or less than N , includes all the rational formulas of De Lagney and Halley, which were separately investigated by them; and yet this general formula is perfectly simple and easy to apply, and easier kept in mind than any one of the said particular formulas. For, in words at length, it is simply this: to $n+1$ times N add $n-1$ times a^n , and to $n-1$ times N add $n+1$ times a^n , then the former sum multiplied by a and divided by the latter sum, will give the root $N^{\frac{1}{n}}$ nearly; or, as the latter sum is to the former sum, so is a , the assumed root, to the required root, nearly. Where it is to be observed that a^n may be taken either greater or less than N , and that the nearer it is to it, the better.

9. By substituting for n , in the general theorem, severally the numbers 2, 3, 4, 5, &c. we shall obtain the following particular theorems, as adapted for the 2d, 3d, 4th, 5th, &c. roots, namely, for the

$$\begin{array}{ll}
 \text{2d or square root,} & \frac{3N + a^2}{N + 3a^2} a = N^{\frac{1}{2}} \\
 \text{3d or cube root,} & \frac{4N + 2a^3}{2N + 4a^3} a, \text{ or } \frac{2N + a^3}{N + 2a^3} a = N^{\frac{1}{3}} \\
 \text{4th root} & \frac{5N + 3a^4}{3N + 5a^4} a = N^{\frac{1}{4}} \\
 \text{5th root} & \frac{6N + 4a^5}{4N + 6a^5} a, \text{ or } \frac{3N + 2a^5}{2N + 3a^5} a = N^{\frac{1}{5}} \\
 \text{6th root} & \frac{7N + 5a^6}{5N + 7a^6} a = N^{\frac{1}{6}} \\
 \text{7th root} & \frac{8N + 6a^7}{6N + 8a^7} a, \text{ or } \frac{4N + 3a^7}{3N + 4a^7} a = N^{\frac{1}{7}} \\
 & \text{\&c.}
 \end{array}$$

10. To exemplify now our formula, let it be first required to extract the square root of 365. Here $N = 365$, $n = 2$; the nearest square is 361, whose root is 19.

$$\text{Hence } 3N + a^2 = 3 \times 365 + 361 = 1456,$$

$$\text{and } N + 3a^2 = 365 + 3 \times 361 = 1448;$$

$$\text{then as } 1448 : 1456 :: 19 : \frac{19 \times 182}{181} = 19 \frac{19}{181} = 19.10497 \text{ \&c.}$$

Again, to approach still nearer, substitute this last found root $\frac{19 \times 182}{181}$ for a , the values of the other letters remaining as before, we

$$\text{have } a^2 = \frac{19^2 \times 182^2}{181^2} = \frac{3458^2}{181^2}; \text{ then}$$

$$3N + a^2 = 3 \times 365 + \frac{3458^2}{181^2} = \frac{47831059}{32761},$$

$$N + 3a^2 = 365 + \frac{3 \times 3458^2}{181^2} = \frac{47831057}{32761};$$

hence $47831057 : 47831059 :: \frac{19 \times 182}{181}$ or $\frac{3458}{181} : \frac{3458 \times 47831059}{181 \times 47831057} =$
the root of 365 very exact, which being brought into decimals, would be true to about 20 places of figures.

11. For a second example, let it be proposed to double the cube, or to find the cube root of the number 2.

Here $N = 2$, $n = 3$, the nearest root $a = 1$, also $a^3 = 1$.

$$\text{Hence } 2N + a^3 = 4 + 1 = 5,$$

$$\text{and } N + 2a^3 = 2 + 2 = 4;$$

$$\text{then as } 4 : 5 :: 1 : \frac{5}{4} = 1.25 = \text{the first approximation.}$$

$$\text{Again, take } a = \frac{5}{4}, \text{ and consequently } a^3 = \frac{125}{64};$$

$$\text{Hence } 2N + a^3 = 4 + \frac{125}{64} = \frac{381}{64},$$

$$\text{and } N + 2a^3 = 2 + \frac{250}{64} = \frac{378}{64};$$

then as $378 : 381$, or as $126 : 127 :: \frac{5}{4} : \frac{5}{4} \times \frac{127}{126} = \frac{635}{504} = 1.259921$,
for the cube root of 2, which is true in the last figure.

And by taking $\frac{635}{504}$ for the value of a , and repeating the process, a great many more figures may be found.

12. For

12. For a third example, let it be required to find the 5th root of 2.

Here $N = 2$, $n = 5$, the nearest root $a = 1$.

$$\text{Hence } 3N + 2a^5 = 6 + 2 = 8,$$

$$\text{and } 2N + 3a^5 = 4 + 3 = 7;$$

then as $7 : 8 :: 1 : \frac{8}{7} = 1\frac{1}{7}$ for the first approximation.

Again, taking $a = \frac{8}{7}$, we have

$$3N + 2a^5 = 6 + \frac{65536}{16807} = \frac{166378}{16807},$$

$$2N + 3a^5 = 4 + \frac{98304}{16807} = \frac{165532}{16807};$$

then as $165532 : 166378 :: \frac{8}{7} : \frac{8}{7} \times \frac{83189}{82766} = \frac{4}{7} \times \frac{83189}{41383} = \frac{332756}{289681} = 1.148698 \text{ \&c.}$ for the 5th root of 2, and is true in the last figure.

13. To find the 7th root of $126\frac{1}{5}$.

Here $N = 126\frac{1}{5}$, $n = 7$, the nearest root $a = 2$, also $a^7 = 128$.

$$\text{Hence } 4N + 3a^7 = 504\frac{4}{5} + 384 = 888\frac{4}{5} = \frac{4444}{5},$$

$$\text{and } 3N + 4a^7 = 378\frac{3}{5} + 512 = 890\frac{3}{5} = \frac{4453}{5};$$

then as $4453 : 4444 :: 2 : \frac{8888}{4453} = 1.995957$, for the root very exact by one operation, being true to the nearest unit in the last figure.

14. To find the 365th root of 1.05, or the amount of 1 pound for 1 day, at 5 per cent. per annum, compound interest.

Here $N = 1.05$, $n = 365$, $a = 1$ the nearest root.

$$\text{Hence } 366N + 364a = 748.3,$$

$$\text{and } 364N + 366a = 748.2;$$

then as $748.2 : 748.3 :: 1 : \frac{7483}{7482} = 1\frac{1}{7482} = 1.00013366$, the root sought very exact at one operation.

15. Let it be required to find the value of the

$$\text{quantity } \sqrt[3]{5\frac{1}{4}} \text{ or } \sqrt[3]{\frac{21}{4}}.$$

Now this may be done two ways; either by finding the $\frac{3}{2}$ power or $\frac{3}{2}$ root of $\frac{21}{4}$ at once; or else by finding the 3d or cubic root of $\frac{21}{4}$, and then squaring the result.

By the first way:—Here it is easy to see that a is nearly $= 3$, because $3^{\frac{3}{2}} = \sqrt{27} = 5 + \text{some small fraction}$. Hence, to find nearly the square root of 27, or $\sqrt{27}$, the nearest power to which is $25 = a^2$ in this case:

$$\text{Hence } 3N + a^2 = 3 \times 27 + 25 = 106,$$

$$\text{and } N + 3a^2 = 27 + 3 \times 25 = 102;$$

then as $102 : 106$, or as $51 : 53 :: 5 : \frac{5 \times 53}{51} = \frac{265}{51} = \sqrt{27}$ nearly.

Then having $N = \frac{21}{4}$, $n = \frac{3}{2}$, $a = 3$, and $a^{\frac{3}{2}} = \frac{265}{51}$ nearly;

$$\text{it will be } \frac{5}{2}N + \frac{1}{2}a^{\frac{3}{2}} = \frac{5}{2} \times \frac{21}{4} + \frac{1}{2} \times \frac{265}{51} = \frac{6415}{408},$$

$$\text{and } \frac{1}{2}N + \frac{5}{2}a^{\frac{3}{2}} = \frac{1}{2} \times \frac{21}{4} + \frac{5}{2} \times \frac{265}{51} = \frac{6371}{408};$$

hence as $6371 : 6415 :: 3 : \frac{19245}{6371} = 3 \frac{134}{6371} = 3.020719$, for the value of the quantity sought nearly, by this way.

Again, by the other method, in finding first the value of $\left(\frac{21}{4}\right)^{\frac{1}{3}}$, or the cube root of $\frac{21}{4}$. It is evident that 2 is the nearest integer root, being the cube root of $8 = a^3$.

$$\text{Hence } 2N + a^3 = \frac{21}{2} + 8 = \frac{74}{4},$$

$$\text{and } N + 2a^3 = \frac{21}{4} + 16 = \frac{85}{4};$$

then as $85 : 74 :: 2 : \frac{148}{85}$, or $= \frac{71}{4}$ nearly. Then taking $\frac{7}{4}$ for

$$a, \text{ we have } 2N + a^3 = \frac{21}{2} + \frac{343}{64} = \frac{1015}{64},$$

$$\text{and } N + 2a^3 = \frac{21}{4} + \frac{2 \cdot 343}{64} = \frac{1022}{64};$$

hence

hence as 1022 : 1015, or as 146 : 145 :: $\frac{7}{4} : \frac{7}{4} \times \frac{145}{146} = \sqrt[21]{\frac{7}{4}}$ nearly.

Consequently the square of this, or $\sqrt[21]{\frac{7}{4}}$ will be $= \frac{7^2}{4^2} \times \frac{145^2}{146^2} = \frac{1030225}{341056}$

$= 3 \frac{7057}{341056} = 3.020690$, the quantity sought more nearly, being true in the last figure.



V.

A new Method of finding, in finite and general Terms, near Values of the Roots of Equations of this Form, $x^n - px^{n-1} + qx^{n-2} \&c. = 0$; namely, having the Terms alternately Plus and Minus.

1. **T**HE following is one method more, to be added to the many we are already possessed of, for determining the roots of the higher equations. By means of it we readily find a root, which is sometimes accurate; and when not so, it is at least near the truth, and that by an easy finite formula, which is general for all equations of the above form, and of the same dimension, provided that root be a real one. This is of use for depressing the equation down to lower dimensions, and thence for finding all the roots one after another, when the formula gives the root sufficiently exact; and when not, it serves as a ready means of obtaining a near value of a root, by which to commence an approximation still nearer, by the previously known methods of Newton, or Halley, or others. This method is farther useful in elucidating the nature of equations, and certain properties of numbers; as will appear in some of the following articles. We have already easy methods for finding the roots of simple and quadratic equations. I shall therefore begin with the cubic equation, and treat of each order of equations separately, in ascending gradually to the higher dimensions.

2. Let then the cubic equation $x^3 - px^2 + qx - r = 0$ be proposed. Assume the root $x = a$, either accurately or approximately, as it may happen, so that $x - a = 0$, accurately or nearly. Raise this

x —

$x - a = 0$ to the third power, the same dimension with the proposed equation, so shall $x^3 - 3ax^2 + 3a^2x - a^3 = 0$;

but the proposed equation is $x^3 - px^2 + qx - r = 0$;

therefore the one of these is equal to the other. But the first term (x^3) of each is the same; and hence, if we assume the second terms equal between themselves, it will follow that the sum of the two remaining terms will also be equal, and give a simple equation by which the value of x is determined. Thus, $3ax^2$ being $= px^2$, or $a = \frac{1}{3}p$, we shall have $3a^2x - a^3 = qx - r$,

and hence $x = \frac{a^3 - r}{3a^2 - q} = \frac{\frac{1}{27}p^3 - r}{3 \times \frac{1}{9}p^2 - q} = \frac{p^3 - 27r}{p^2 - 3q} \times \frac{1}{9}$, by substituting $\frac{1}{3}p$, the value of a , instead of it.

3. Now this value of x here found, will be the middle root of the proposed cubic equation. For because a is assumed nearly or accurately equal to x , and also equal to $\frac{1}{3}p$, therefore x is $= \frac{1}{3}p$ nearly or accurately, that is, $\frac{1}{3}$ of the sum of the three roots, to which the coefficient p of the second term of the equation, is always equal; and thus, being a medium among the three roots, it will be either nearly or accurately equal to the middle root of the proposed equation, when that root is a real one.

4. Now this value of x will always be the middle root *accurately*, whenever the three roots are in arithmetical progression; otherwise, only *approximately*. For when the three roots are in arithmetical progression, $\frac{1}{3}p$ or $\frac{1}{3}$ of their sum, it is well known, is equal to the middle term or root. In the other cases, therefore, the above-found value of x is only *near* the middle root.

5. When the roots are in arithmetical progression, because the middle term or root is then $= \frac{1}{3}p$, and also $= \frac{1}{9} \times \frac{p^3 - 27r}{p^2 - 3q}$, therefore $\frac{1}{3}p = \frac{1}{9} \times \frac{p^3 - 27r}{p^2 - 3q}$, or $2p^3 = 9pq - 27r = 9 \times \frac{p^3 - 27r}{p^2 - 3q}$, an equation

tion expressing the general relation of p , q , and r ; where p is the sum of any three terms in arithmetical progression, q the sum of their three rectangles, and r the product of all the three. For, in any equation, the coefficient p of the second term, is the sum of the roots; the coefficient q of the third term, is the sum of the rectangles of the roots; and the coefficient r of the fourth term, is the sum of the solids of the roots, which in the case of the cubic equation is only one:—Thus, if the roots, or arithmetical terms, be 1, 2, 3. Here $p = 1 + 2 + 3 = 6$, $q = 1 \times 2 + 1 \times 3 + 2 \times 3 = 2 + 3 + 6 = 11$, $r = 1 \times 2 \times 3 = 6$; then $2p^3 = 2 \times 6^3 = 432$, and $9 \times \overline{pq} - 3r = 9 \times 48 = 432$ also.

6. To illustrate now the rule $x = \frac{1}{9} \times \frac{p^3 - 27r}{p^2 - 3q}$ by some examples; let us in the first place take the equation $x^3 - 6x^2 + 11x - 6 = 0$. Here $p = 6$, $q = 11$, and $r = 6$; consequently

$$x = \frac{1}{9} \times \frac{p^3 - 27r}{p^2 - 3q} = \frac{1}{9} \times \frac{6^3 - 27 \times 6}{6^2 - 3 \times 11} = \frac{8 - 6}{12 - 11} = \frac{2}{1} = 2.$$

This being substituted for x in the given equation, makes all the terms to vanish, and therefore it is an exact root, and the roots will be in arithmetical progression. Dividing therefore the given equation by $x - 2 = 0$, the quotient is $x^2 - 4x + 3 = 0$, the roots of which quadratic equation are 3 and 1, the other two roots of the proposed equation $x^3 - 6x^2 + 11x - 6 = 0$.

7. If the equation be $x^3 - 39x^2 + 479x - 1881 = 0$; we shall have $p = 39$, $q = 479$, and $r = 1881$; then $x =$

$$\frac{1}{9} \times \frac{p^3 - 27r}{p^2 - 3q} = \frac{1}{9} \times \frac{39^3 - 27 \times 1881}{39^2 - 3 \times 479} = \frac{13^3 - 1881}{13^2 - 3 \times 479} = \frac{316}{28} = \frac{79}{7} = 11\frac{2}{7}.$$

Then, substituting $11\frac{2}{7}$ for x in the proposed equation, the negative terms are found to exceed the positive terms by 5, thereby shewing that $11\frac{2}{7}$ is very near, but something above, the middle root, and that therefore the roots are not in arithmetical progression. It is therefore probable

bable that 11 may be the true value of the root, and on trial it is found to succeed,

Then dividing $x^3 - 39x^2 + 479x - 1881$ by $x - 11$, the quotient is $x^2 - 28x + 171 = 0$, the roots of which quadratic equation are 9 and 19, the two other roots of the proposed equation.

8. If the equation be $x^3 - 6x^2 + 9x - 2 = 0$;
we shall have $p = 6$, $q = 9$, and $r = 2$; then $x =$

$$\frac{1}{9} \times \frac{p^3 - 27r}{p^2 - 3q} = \frac{1}{9} \times \frac{6^3 - 27 \times 2}{6^2 - 3 \times 9} = \frac{2^3 - 2}{12 - 9} = \frac{6}{3} = 2.$$

This value of x being substituted for it in the proposed equation, causes all the terms to vanish, as it ought, thereby shewing that 2 is the middle root, and that the roots are in arithmetical progression.

Accordingly, dividing the given quantity $x^3 - 6x^2 + 9x - 2$ by $x - 2$, the quotient is $x^2 - 4x + 1 = 0$, a quadratic equation, whose roots are $2 + \sqrt{2}$ and $2 - \sqrt{2}$, the two other roots of the equation proposed,

9. If the equation be $x^3 - 5x^2 + 5x - 1 = 0$;
we shall have $p = 5$, $q = 5$, and $r = 1$; then $x =$

$$\frac{1}{9} \times \frac{5^3 - 27 \times 1}{5^2 - 3 \times 5} = \frac{1}{9} \times \frac{125 - 27}{25 - 15} = \frac{1}{9} \times \frac{98}{10} = \frac{49}{45} = 1\frac{4}{9}.$$

From which one might guess the root ought to be 1, and which being tried, is found to succeed.

But without such trial, we may make use of this value $1\frac{4}{9}$, or $1\frac{1}{2}$ nearly, and approximate with it in the common way.

Having found the middle root to be 1, divide the given quantity $x^3 - 5x^2 + 5x - 1$ by $x - 1$, and the quotient is $x^2 - 4x + 1 = 0$, the roots of which are $2 + \sqrt{2}$ and $2 - \sqrt{2}$, the two other roots, as in the last article.

10. If the equation be $x^3 - 7x^2 + 18x - 18 = 0$;
we shall have $p = 7$, $q = 18$, and $r = 18$; then $x =$

$$\frac{1}{9} \times \frac{7^3 - 27 \times 18}{7^2 - 3 \times 18} = \frac{1}{9} \times \frac{343 - 486}{49 - 54} = \frac{143}{45} = 3\frac{8}{45} \text{ or } 3 \text{ nearly.}$$

Then trying 3 for x , it is found to succeed. And dividing $x^3 - 7x^2 + 18x - 18$ by $x - 3$, the quotient is $x^2 - 4x + 6 = 0$, a quadratic equation whose roots are $2 + \sqrt{-2}$ and $2 - \sqrt{-2}$, the two other roots of the proposed equation, which are both impossible or imaginary.

11. If the equation be $x^3 - 6x^2 + 14x - 12 = 0$;
we shall have $p = 6$, $q = 14$, and $r = 12$; then $x =$

$\frac{1}{9} \times \frac{6^3 - 27 \times 12}{6^2 - 3 \times 14} = \frac{1}{9} \times \frac{216 - 324}{36 - 42} = \frac{108}{54} = 2$. Which being substituted for x , it is found to answer, the sum of the terms coming out $= 0$. Therefore the roots are in arithmetical progression. And, accordingly, by dividing $x^3 - 6x^2 + 14x - 12$ by $x - 2$, the quotient is $x^2 - 4x + 6 = 0$, the roots of which quadratic equation are $2 + \sqrt{-2}$ and $2 - \sqrt{-2}$, the two other roots of the proposed equation, and the common difference of the three roots is $\sqrt{-2}$.

12. But if the equation be $x^3 - 8x^2 + 22x - 24 = 0$;
we shall have $p = 8$, $q = 22$, and $r = 24$; then $x =$

$\frac{1}{9} \times \frac{8^3 - 27 \times 24}{8^2 - 3 \times 22} = \frac{1}{9} \times \frac{512 - 648}{64 - 66} = \frac{136}{18} = \frac{68}{9} = 7\frac{5}{9}$. Which being substituted for x in the proposed equation, the sum of the terms differs very widely from the truth, thereby shewing that the middle root of the equation is an imaginary one, as it is indeed, the three roots being 4, and $2 + \sqrt{-2}$, and $2 - \sqrt{-2}$.

13. In Art. 2 the value of x was determined by assuming the second terms of the two equations equal to each other. But a like near value might be determined by assuming either the two third terms, or the two fourth terms equal.

Thus

Thus the equations being $\begin{cases} x^3 - 3ax^2 + 3a^2x - a^3 = 0, \\ x^3 - px^2 + qx - r = 0, \end{cases}$
if we assume the third terms $3a^2x$ and qx equal, or $a = \sqrt[3]{\frac{1}{3}q}$, the sums of the second and fourth terms will be equal, namely, $3ax^2 + a^3 = px^2 + r$; and hence we find

$$x = \sqrt{\frac{a^3 - r}{p - 3a}} = \sqrt{\frac{\sqrt[3]{\frac{1}{3}q}^3 - r}{p - 3\sqrt[3]{\frac{1}{3}q}}},$$

by substituting $\sqrt[3]{\frac{1}{3}q}$ the value of a instead of it.

And if we assume the fourth terms equal, namely $a^3 = r$, or $a = \sqrt[3]{r}$, then the sums of the second and third terms will be equal, namely, $3ax - 3a^2 = px - q$; and hence $x = \frac{q - 3a^2}{p - 3a} = \frac{q - 3r^{\frac{2}{3}}}{p - 3r^{\frac{1}{3}}}$, by substituting $r^{\frac{1}{3}}$ instead of a . And either of these two formulas will give nearly the same value of the root as the first formula, at least when the roots do not differ very greatly from one another.

But if they differ very much among themselves, the first formula will not be so accurate as these two others, because that in them the roots were more complexly mixed together; for the second formula is drawn from the coefficient of the third term, which is the sum of all the rectangles of the roots; and the third formula is drawn from the coefficient of the last term, which is equal to the continual product of all the roots; while the first formula is drawn from the coefficient of the second term, which is simply the sum of the roots. And indeed the last theorem is commonly the nearest of all. So that when we suspect the roots to be very wide of each other, let either the second or third be used.

Thus, in the equation $x^3 - 23x^2 + 62x - 40 = 0$, whose three roots are 1, 2, and 20. Here $p = 23$, $q = 62$, $r = 40$; and by the

$$1^{\text{st}} \text{ theor. } x = \frac{1}{9} \times \frac{23^3 - 27 \times 40}{23^2 - 3 \times 62} = \frac{1}{9} \times \frac{12167 - 1080}{529 - 186} = 3\frac{3}{5} \text{ nearly,}$$

$$2^{\text{d}} \text{ theor. } x = \sqrt{\frac{\left(\frac{62}{3}\right)^{\frac{3}{2}} - 40}{23 - 3\sqrt{\frac{62}{3}}}} = \sqrt{\frac{94 - 40}{23 - 12.87}} = \sqrt{5.34} = 2\frac{1}{3} \text{ nearly,}$$

$$3^{\text{d}} \text{ theor. } x = \frac{62 - 3 \times 40^{\frac{2}{3}}}{23 - 3 \times 40^{\frac{1}{3}}} = \frac{62 - 35.1}{26 - 10\frac{1}{4}} = \frac{12}{7} = 1\frac{5}{7} \text{ nearly.}$$

Where the two latter are much nearer the middle root (2) than the first.

And the mean between these two is $2\frac{1}{4}$, which is very near to that root. And this is commonly the case; the one being nearly as much too great as the other is too little.

14. To proceed now, in like manner, to the biquadratic equation, which is of this general form

$$x^4 - px^3 + qx^2 - rx + s = 0.$$

Assume the root $x = a$, or $x - a = 0$, and raise this equation $x - a = 0$ to the fourth power, or the same height with the proposed equation, which will give $x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4 = 0$; but the proposed equation is $x^4 - px^3 + qx^2 - rx + s = 0$; therefore these two are equal to each other. Now if we assume the second terms equal, namely $4a = p$, or $a = \frac{1}{4}p$, then the sums of the three remaining terms will also be equal, namely,

$$6a^2x^2 - 4a^3x + a^4 = qx^2 - rx + s; \text{ and hence}$$

$$\overline{6a^2 - q} \cdot x^2 - \overline{4a^3 - r} \cdot x = s - a^4, \text{ or}$$

$\frac{3}{8}p^2 - q \cdot x^2 - \frac{1}{16}p^3 - r \cdot x = s - \frac{1}{256}p^4$ by substituting $\frac{1}{4}p$ instead of a : then, resolving this quadratic equation, we find its roots to be thus

$$x = \frac{p^3 - 16r \pm \sqrt{p^3 - 16r^2 - \frac{3}{2}p^2 - 4q \cdot p^4 - 256s}}{8 \times \frac{3}{2}p^2 - 4q};$$

$$\text{or if we put } A = \frac{3}{2}p^2 - 4q,$$

$$B = p^3 - 16r,$$

$$C = p^4 - 256s,$$

$$\text{the two roots will be } x = \frac{B \pm \sqrt{B^2 - AC}}{8A}.$$

15. It is evident that the same property is to be understood here, as for the cubic equation in Art. 3, namely, that the two roots above found, are the middle roots of the four which belong to the biquadratic equation, when those roots are real ones; for otherwise the formulæ are

of no use. But however those roots will not be accurate, when the sum of the two middle roots, of the proposed equation, is equal to the sum of the greatest and least roots, or when the four roots are in arithmetical progression; because that, in this case, $\frac{1}{4}p$, the assumed value of a , is neither of the middle roots exactly, but only a mean between them.

16. To exemplify this formula $x = \frac{B \pm \sqrt{B^2 + AC}}{8A}$, let the proposed equation be $x^4 - 12x^3 + 49x^2 - 78x + 40 = 0$. Then

$$A = \frac{3}{2}p^2 - 4q = 12^2 \times \frac{3}{2} - 4 \times 49 = 216 - 196 = 20,$$

$$B = p^3 - 16r = 12^3 - 16 \times 78 = 1728 - 1248 = 480,$$

$$C = p^4 - 256s = 12^4 - 256 \times 40 = 20736 - 10240 = 10496.$$

Hence $x = \frac{B \pm \sqrt{B^2 - AC}}{8A} = \frac{480 \pm \sqrt{480^2 - 20 \times 10496}}{8 \times 20} = \frac{15 \pm \sqrt{40}}{5} = 3 \pm 1\frac{1}{4}$ nearly, or $4\frac{1}{4}$ and $1\frac{3}{4}$ nearly, or nearly 4 and 2, whose sum is 6. And trying 4 and 2, they are both found to answer, and therefore they are the two middle roots.

Then $\overline{x - 4} \times \overline{x - 2} = x^2 - 6x + 8$, by which dividing the given equation $x^4 - 12x^3 + 49x^2 - 78x + 40 = 0$, the quotient is $x^2 - 6x + 5 = 0$, the roots of which quadratic equation are 5 and 1, and which therefore are the greatest and least roots of the equation proposed.

17. If the equation be $x^4 - 12x^3 + 47x^2 - 72x + 36 = 0$; then

$$A = \frac{3}{2}p^2 - 4q = 12^2 \times \frac{3}{2} - 4 \times 47 = 216 - 188 = 28,$$

$$B = p^3 - 16r = 12^3 - 16 \times 72 = 1728 - 1152 = 576,$$

$$C = p^4 - 256s = 12^4 - 256 \times 36 = 20736 - 9216 = 11520.$$

Hence $x = \frac{B \pm \sqrt{B^2 - AC}}{8A} = \frac{576 \pm \sqrt{576^2 - 28 \times 11520}}{8 \times 28} = \frac{18 \pm 3}{7} = 3$ and $2\frac{1}{7}$, or 3 and 2 nearly; both of which answer on trial; and therefore 3 and 2 are the two middle roots.

Then

Then $\overline{x-3} \times \overline{x-2} = x^2 - 5x + 6 = 0$, by which dividing the given quantity $x^4 - 12x^3 + 47x^2 - 72x + 36 = 0$, the quotient is $x^2 - 7x + 6 = 0$, the roots of which quadratic equation are 6 and 1, which therefore are the greatest and least roots of the equation proposed.

18. If the equation be $x^4 - 7x^3 + 15x^2 - 11x + 3 = 0$; we have $A = \frac{3}{2}p^2 - 4q = 7^2 \times \frac{3}{2} - 4 \times 15 = 73\frac{1}{2} - 60 = 13\frac{1}{2}$,
 $B = p^3 - 16r = 7^3 - 16 \times 11 = 343 - 176 = 167$,
 $C = p^4 - 256s = 7^4 - 256 \times 3 = 2401 - 768 = 1633$.

Hence $x = \frac{B \pm \sqrt{B^2 - AC}}{8A} = \frac{167 \pm \sqrt{167^2 - 13\frac{1}{2} \times 1633}}{8 \times 13\frac{1}{2}} = \frac{167 \pm 76}{108} = 2\frac{1}{3}$ and $\frac{9}{10}$ nearly, or nearly 2 and 1; both which are found, on trial, to answer; and therefore 2 and 1 are the two middle roots sought.

Then $\overline{x-2} \times \overline{x-1} = x^2 - 3x + 2$, by which dividing the given equation $x^4 - 7x^3 + 15x^2 - 11x + 3 = 0$, the quotient is $x^2 - 4x + 1 = 0$, the roots of which quadratic equation are $2 + \sqrt{2}$ and $2 - \sqrt{2}$, and which therefore are the greatest and least roots of the proposed equation.

19. But if the equation be $x^4 - 9x^3 + 30x^2 - 46x + 24 = 0$; we have $A = \frac{3}{2}p^2 - 4q = 9^2 \times \frac{3}{2} - 4 \times 30 = 121\frac{1}{2} - 120 = 1\frac{1}{2}$,
 $B = p^3 - 16r = 9^3 - 16 \times 46 = 729 - 736 = -7$,
 $C = p^4 - 256s = 9^4 - 256 \times 24 = 6561 - 6144 = 417$.

Hence $x = \frac{B \pm \sqrt{B^2 - AC}}{8A} = \frac{-7 \pm \sqrt{49 - 625\frac{1}{2}}}{8 \times 1\frac{1}{2}} = \frac{-7 \pm \sqrt{-576\frac{1}{2}}}{12}$, an imaginary quantity, shewing that the two middle roots are imaginary, and therefore the formula is of no use in this case, the four roots being 1, $2 + \sqrt{-2}$, $2 - \sqrt{-2}$, and 4.

20. And thus in other examples the two middle roots will be found when they are rational, or a near value when irrational, which in this case

case will serve for the foundation of a nearer approximation, to be made in the usual way.

We might also find another formula for the biquadratic equation, by assuming the last terms as equal to each other; for then the sum of the 2d, 3d, and 4th terms of each would be equal, and would form another quadratic equation, whose roots would be nearly the two middle roots of the biquadratic proposed.

21. Or a root of the biquadratic equation may easily be found, by assuming it equal to the product of two squares, as $\overline{x-a}^2 \times \overline{x-b}^2 = x^4 - 2 \cdot \overline{a+b} \cdot x^3 + \overline{2ab+a+b} \cdot x^2 - 2ab \cdot \overline{a+b} \cdot x + a^2 b^2 = 0$. For, comparing the terms of this with the terms of the equation proposed, in this manner, namely, making the second terms equal, then the third terms equal, and lastly the sums of the fourth and fifth terms equal, these equations will determine a near value of x by a simple equation. For those equations are

$$p = 2 \cdot \overline{a+b}, \text{ or } \frac{1}{2}p = \overline{a+b},$$

$$q = \overline{2ab + a + b}^2 = 2ab + \frac{1}{4}p^2, \text{ or } 2ab = q - \frac{1}{4}p^2,$$

$$rx - s = 2ab \cdot \overline{a+b} \cdot x - a^2 b^2 = \frac{1}{2}p \cdot \overline{q - \frac{1}{4}p^2} \cdot x - \frac{1}{4} \cdot \overline{q - \frac{1}{4}p^2}^2.$$

Then the values of ab and $a+b$, found from the first and second of these equations, and substituted in the third, this gives

$$x = \frac{s - \frac{1}{2}p \cdot \overline{q - \frac{1}{4}p^2}}{r - p \cdot \frac{1}{2} \cdot \overline{q - \frac{1}{4}p^2}} = \frac{64s - 4p \cdot \overline{q - p^2}}{64r - 8p \cdot \overline{4q - p^2}}, \text{ a general formula for one of the roots of the biquadratic equation } x^4 - px^3 + qx^2 - rx + s = 0.$$

22. To exemplify now this formula, let us take the same equation as in Art. 17, namely, $x^4 - 12x^3 + 47x^2 - 72x + 36 = 0$, the roots of which were there found to be 1, 2, 3, and 6. Then, by our last formula we shall have

$$x = \frac{64s - 4p \cdot \overline{q - p^2}}{64r - 8p \cdot \overline{4q - p^2}} = \frac{64 \times 36 - 4 \times 47 - 12^2}{64 \times 72 - 96 \times 4 \times 47 - 12^2} = \frac{64 \times 36 - 44 \times 44}{64 \times 72 - 96 \times 44} = \frac{23}{24},$$

or nearly 1, which is the least root.

23. Again,

23. Again, in the equation $x^4 - 7x^3 + 15x - 11x^2 + 3 = 0$, whose roots are 1, 2, $2 + \sqrt{2}$, and $2 - \sqrt{2}$, we have

$$x = \frac{64 \times 3 - \overline{60 - 49}^2}{64 \times 11 - 56 \times \overline{60 - 49}} = \frac{64 \times 3 - 11 \times 11}{64 \times 11 - 56 \times 11} = \frac{192 - 121}{704 - 616} = \frac{71}{88} = \frac{4}{5}$$

nearly, which is nearly a mean between the two least roots 1 and $2 - \sqrt{2}$ or $\frac{3}{5}$ nearly.

24. But if the equation be

$$x^4 - 9x^3 + 30x^2 - 46x + 24 = 0,$$

which has impossible roots, the four roots being 1, $2 + \sqrt{-2}$, $2 - \sqrt{-2}$, and 4; we shall have

$$x = \frac{64 \times 24 - \overline{120 - 81}^2}{64 \times 46 - 72 \times \overline{120 - 81}} = \frac{64 \times 24 - 39 \times 39}{64 \times 46 - 72 \times 39} = \frac{1536 - 1521}{2944 - 2808} = \frac{15}{136} = \frac{1}{9}$$

nearly, which is of no use in this case of imaginary roots.

25. This formula will also sometimes fail when the roots are all real. As if the equation be

$$x^4 - 12x^3 + 49x^2 - 78x + 40 = 0,$$

the roots of which are 1, 2, 4, and 5. For here

$$\begin{aligned} x &= \frac{64 \times 40 - \overline{196 - 144}^2}{64 \times 78 - 96 \times \overline{196 - 144}} = \frac{64 \times 40 - 52 \times 52}{64 \times 78 - 96 \times 52} = \frac{16 \times 10 - 13 \times 13}{16 \times 19\frac{1}{2} - 24 \times 13} \\ &= \frac{160 - 169}{312 - 312} = \frac{-9}{0} 7, \text{ which is of no use.} \end{aligned}$$

26. For equations of higher dimensions, as the 5th, the 6th, the 7th, &c. we might, in imitation of this last method, combine other forms of quantities together. Thus, for the 5th power, we might compare it either with $\overline{x - a}^4 \times \overline{x - b}$, or with $\overline{x - a}^3 \times \overline{x - b}^2$, or with $\overline{x - a}^3 \times \overline{x - b} \times \overline{x - c}$, or with $\overline{x - a}^2 \times \overline{x - b}^2 \times \overline{x - c}$. And so for the other powers.



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VI.

Of the Binomial Theorem. With a Demonstration of the Truth of it in the General Case of Fractional Exponents.

1. **I**T is well known that this famous theorem is called *binomial*, because it contains a proposition of a quantity consisting of *two* terms, as a radix, to be expanded in a series of equal value. It is also called emphatically the Newtonian theorem, or Newton's binomial theorem, because he has commonly been reputed the author of it, as he was indeed for the case of fractional exponents, which is the most general of all, and includes all the other particular cases, of powers, or divisions, &c.

2. The binomial, as proposed in its general form, was, by Newton, thus expressed $P + PQ^{\frac{m}{n}}$; where P is the first term of the binomial, Q the quotient of the second term divided by the first, and consequently PQ is the second term itself; or PQ may represent all the terms of a multinomial, after the first term, and consequently Q the quotient of all those terms, except the first term, divided by that first term, and may be either positive or negative; also $\frac{m}{n}$ represents the exponent of the binomial, and may denote any quantity, integral or fractional, positive or negative, rational or furd. When the exponent

is integral, the denominator n is equal to 1, and the quantity then in this form $\overline{P + PQ}^m$, denotes a binomial to be raised to some power; the series for which was fully determined before Newton's time, as I have shewn in the historical introduction to my Mathematical Tables, lately published. When the exponent is fractional, m and n may be any quantities whatever, m denoting the index of some power to which the binomial is to be raised, and n the index of the root to be extracted of that power: and to this case it was first extended and applied by Newton. When the exponent is negative, the reciprocal

of the same quantity is meant; as $\overline{P + PQ}^{-\frac{m}{n}}$ is equal to $\frac{1}{\overline{P + PQ}^{\frac{m}{n}}}$.

3. Now when the radical binomial is expanded in an equivalent series, it is asserted that it will be in this general form, namely

$$\begin{aligned} & \overline{P + PQ}^{\frac{m}{n}} \text{ or } P^{\frac{m}{n}} \times \overline{1 + Q}^{\frac{m}{n}} \\ &= P^{\frac{m}{n}} \times 1 + \frac{m}{n} Q + \frac{m}{n} \cdot \frac{m-n}{2n} Q^2 + \frac{m}{n} \cdot \frac{m-2n}{3n} \cdot \frac{m-3n}{4n} Q^3 + \&c. \\ & \text{or} \\ &= P^{\frac{m}{n}} \times 1 + \frac{m}{n} A Q + \frac{m-n}{2n} B Q + \frac{m-3n}{4n} C Q + \frac{m-4n}{5n} D Q + \&c. \end{aligned}$$

where the law of the progression is visible, and the quantities P, m, n, Q , include their signs $+$ or $-$, the terms of the series being all positive when Q is positive, and alternately positive and negative when Q is negative, independent however of the effect of the coefficients made up of m and n : also $A, B, C, D, \&c.$ in the latter form, denote each preceding term. This latter form is the easier in practice, when we want

to

to collect the sum of the terms of a series; but the former is the fitter for shewing the law of the progression of the terms.

4. The truth of this series was not demonstrated by Newton, but only inferred by way of induction. Since his time however, several attempts have been made to demonstrate it, with various success, and in various ways; of which however those are justly preferred, which proceed by pure algebra, and without the help of fluxions. And such has been esteemed the difficulty of proving the general case independent of the doctrine of fluxions, that many eminent mathematicians to this day account the demonstration not fully accomplished, and still a thing greatly to be desired. Such a demonstration I think I have effected. But before I deliver it, it may not be improper to premise somewhat of the history of this theorem, its rise, progress, extension, and demonstrations.

5. Till very lately the prevailing opinion has been, that the theorem was not only invented by Newton, but first of all by him; that is, in that state of perfection in which the terms of the series for any assigned power whatever, can be found independently of the terms of the preceding powers; namely, the second term from the first, the third term from the second, the fourth term from the third, and so on, by a general rule. Upon this point I have already given an opinion in the history to my logarithms, above cited, and I shall here enlarge somewhat farther on the same head.

That Newton invented it himself, I make no doubt. But that he was not the first inventor, is at least as certain. It was described by Briggs, in his *Trigonometria Britannica*, long before Newton was born; not indeed for fractional exponents, for that was the application of Newton, but for any integral power whatever, and that by the general law of the terms as laid down by Newton, independent of the terms of the powers preceding that which is required. For as to the generation of the coefficients of the terms of one power from those of

the preceding powers, successively one after another, it was remarked by Vieta, Oughtred, and many others, and was not unknown to much more early writers on arithmetic and algebra, as will be manifest by a slight inspection of their works, as well as the gradual advance the property made, both in extent and perspicuity, under the hands of the successive masters in arithmetic, every one adding somewhat more towards the perfection of it.

6. Now the knowledge of this property of the coefficients of the terms in the powers of a binomial, is at least as old as the practice of the extraction of roots; for this property was both the foundation, the principle, and the means of those extractions. And as the writers on arithmetic became acquainted with the nature of the coefficients in powers still higher, just so much higher did they extend the extraction of roots, still making use of this property. At first it seems they were only acquainted with the nature of the square, which consists of these three terms, 1, 2, 1; and accordingly extracted the square roots of numbers by means of them; but went no farther. The nature of the cube next presented itself, which consists of these four terms, 1, 3, 3, 1; and by means of these they extracted the cubic roots of numbers, in the same manner as we do at present. And this was the extent of their extractions in the time of Lucas de Burgo, an Italian, who, from 1470 to 1500, wrote several tracts on arithmetic, containing the sum of what was then known of this science, which chiefly consisted in the doctrine of the proportions of numbers, the nature of figurate numbers, and the extraction of roots, as far as the cubic root inclusively.

7. It was not long however before the nature of the coefficients of all the higher powers became known, and tables formed for constructing them indefinitely. For in the year 1543 came out, at Norimberg, an excellent treatise of arithmetic and algebra, by Michael Stifelius, a German divine, and an honest, but a weak, disciple of Luther. In this work, *Arithmetica Integra*, of Stifelius, are contained several curious things,

things, some of which have been ascribed to a much later date. He here treats pretty fully and ably, of progression and figurate numbers, and in particular of the following table for constructing both them and the coefficients of the terms of all powers of a binomial, which has been so often used since his time for these and other purposes, and which more than a century after was, by Pascal, otherwise called the arithmetical triangle, and who only mentioned some additional properties of the table.

1								
2								
3	3							
4	6							
5	10	10						
6	15	20						
7	21	35	35					
8	28	56	70					
9	36	84	126	126				
10	45	120	210	252				
11	55	165	330	462	462			
12	66	220	495	792	924			
13	78	286	715	1287	1716	1716		
14	91	364	1001	2002	3003	3432		
15	105	455	1365	3003	5005	6435	6435	
16	120	560	1820	4368	8008	11440	12870	
17	136	680	2380	6188	12376	19448	24310	

Stifelius here observes that the horizontal lines of this table furnish the coefficients of the terms of the correspondent powers of a binomial; and teaches how to use them in extracting the roots of all powers whatever. And after him the same table was used for the same purpose, by Cardan, and Stevin, and the other writers on arithmetic. I suspect, however, that the nature of this table was known much earlier than the time of Stifelius, at least so far as regards the progressions of figurate numbers, a doctrine amply treated of by Nicomachus, who lived, according to some, before Euclid, but not till long after him according to others; and whose work on arithmetic was published at Paris in 1538; and which it is supposed was chiefly copied in the treatise on the same subject by Boethius: but I have never seen either
of

of these two works. Though indeed Cardan seems to ascribe the invention of the table to Stifelius; but I suppose that is only to be understood of its application to the extraction of roots. See Cardan's *Opus Novum de Proportionibus*, where he quotes it, and extracts the table and its use from Stifelius's book. Cardan also, at page 185, *et seq.* of the same work, makes use of a like table to find the number of variations of things, or conjugations as he calls them.

8. The contemplation of this table has probably been attended with the invention and extension of some of our most curious discoveries in mathematics, both in regard to the powers of a binomial, with the consequent extraction of roots, the doctrine of angular sections by Vieta, and the differential method by Briggs and others. For, one or two of the powers or sections being once known, the table would be of excellent use in discovering and constructing the rest. And accordingly we find this table used on many occasions by Stifelius, Cardan, Stevin, Vieta, Briggs, Oughtred, Mercator, Pascal, &c. &c.

9. On this occasion I cannot help mentioning the ample manner in which I see Stifelius, at fol. 35, *et seq.* of the same book, treats of the nature and use of logarithms, though not under the same name, but under the idea of a series of arithmeticals, adapted to a series of geometricals. He there explains all their uses; such as that the addition of them, answers to the multiplication of their geometricals; subtraction to division; multiplication of exponents, to involution; and dividing of exponents, to evolution. And he exemplifies the use of them in cases of the Rule-of-Three, and in finding mean proportionals between given terms, and such like, exactly as is done in logarithms. So that he seems to have been in the full possession of the idea of logarithms, and wanted only the necessity of troublesome calculations to induce him to make a table of such numbers.

10. But

10. But although the nature and conſtruction of this table, namely of figurate numbers, was thus early known, and employed in the raiſing of powers, and extracting of roots; yet it was only by raiſing the numbers one from another by continual additions, and then taking them from the table for uſe when wanted; till Briggs firſt pointed out the way of raiſing any horizontal line in the foregoing table by itſelf, without any of the preceding lines; and thus teaching to raiſe the terms of any power of a binomial, independent of any other powers; and ſo gave the ſubſtance of the binomial ſeries in words, wanting only the notation in ſymbols; as I have ſhewn at large at page 75 of the hiſtorical introduction to my Mathematical Tables.

11. Whatever was known however of this matter, related only to pure or integral powers, no one before Newton having thought of extracting roots by infinite ſeries. He happily diſcovered, that, by conſidering powers and roots in a continued ſeries, roots being as powers having fractional exponents, the ſame binomial ſeries would equally ſerve for them all, whether the index ſhould be fractional or integral, or the ſeries be finite or infinite.

12. The truth of this method however was long known only by trial in particular caſes, and by induction from analogy. Nor does it appear that even Newton himſelf ever attempted any direct proof of it. But various demonſtrations of this theorem have been ſince given by the more modern mathematicians, of which ſome are by means of the doctrine of fluxions, and others, more legally, from the pure principles of algebra only. Some of which I ſhall here give a ſhort account of.

13. One of the firſt was Mr. James Bernoulli. His demonſtration is, among ſeveral other curious things, contained in his little work called *Ars Conjeſtandi*, which has been improperly omitted in the collection of his works published by his nephew Nicholas Bernoulli. This is a ſtrict demonſtration

demonstration of the binomial theorem in the case of integral and affirmative powers, and is to this effect. Supposing the theorem to be true in any one power, as for instance, in the cube, it must be true in the next higher power; which he demonstrates. But it is true in the cube, in the fourth, fifth, sixth, and seventh powers, as will easily appear by trial, that is by actually raising those powers by continual multiplications. Therefore it is true in all higher powers. All this he shews in a regular and legitimate manner, from the principles of multiplication, and without the help of fluxions. But he could not extend his proof to the other cases of the binomial theorem, in which the powers are fractional. And this demonstration has been copied by Mr. John Stewart, in his commentary on Sir Isaac Newton's quadrature of curves. To which he has added, from the principles of fluxions, a demonstration of the other case, for roots or fractional exponents.

14. In No. 230 of the Philosophical Transactions for the year 1697, is given a theorem, by Mr. De Moivre, in imitation of the binomial theorem, which is extended to any number of terms, and thence called the multinomial theorem; which is a general expression in a series, for raising any multinomial quantity to any power. His demonstration of the truth of this theorem, is independent of the truth of the binomial theorem, and contains in it a demonstration of the binomial theorem as a subordinate proposition, or particular case of the other more general theorem. And this demonstration may be considered as a legitimate one, for pure powers, founded on the principles of multiplication, that is, on the doctrine of combinations and permutations. And it proves that the law of the continuation of the terms, must be the same in the terms not computed, or not set down, as in those that are written down.

15. The ingenious Mr. Landen has given an investigation of the binomial theorem, in his *Discourse concerning the Residual Analysis*, printed in 1758, and in the *Residual Analysis* itself, printed in 1764.

The investigation is deduced from this lemma, namely, if m and n be any integers, and $q = \frac{v}{x}$, then is

$$\frac{x^{\frac{m}{n}} - v^{\frac{m}{n}}}{x - v} = x^{\frac{m}{n}-1} \times \frac{1 + q + q^2 + q^3 - \dots - (m)}{1 + q^{\frac{m}{n}} + q^{\frac{2m}{n}} + q^{\frac{3m}{n}} - \dots - (n)}$$

which theorem is made the principal basis of his Residual Analysis.

The investigation is this: the binomial proposed being $1 + x^{\frac{m}{n}}$, assume it equal to the following series $1 + ax + bx^2 + cx^3 \&c.$ with indeterminate coefficients. Then for the same reason

$$\text{as } 1 + x^{\frac{m}{n}} = 1 + ax + bx^2 + cx^3 \&c.$$

$$\text{will } 1 + y^{\frac{m}{n}} = 1 + ay + by^2 + cy^3 \&c.$$

Then, by subtraction,

$$1 + x^{\frac{m}{n}} - 1 + y^{\frac{m}{n}} = a \cdot x - y + b \cdot x^2 - y^2 + c \cdot x^3 - y^3 \&c.$$

And, dividing both sides by $x - y$, and by the lemma, we have

$$\frac{1 + x^{\frac{m}{n}} - 1 + y^{\frac{m}{n}}}{x - y} = \frac{1 + x^{\frac{m}{n}-1}}{1 + x} \times \frac{1 + \frac{1+y}{1+x} + \left(\frac{1+y}{1+x}\right)^2 + \left(\frac{1+y}{1+x}\right)^3 - \dots - (m)}{1 + \left(\frac{1+y}{1+x}\right)^{\frac{m}{n}} + \left(\frac{1+y}{1+x}\right)^{\frac{2m}{n}} + \left(\frac{1+y}{1+x}\right)^{\frac{3m}{n}} - \dots - (n)}$$

$$= a + b \cdot x + y + c \cdot x^2 + xy + y^2 + d \cdot x^3 + x^2y + xy^2 + y^3 \&c.$$

Then, as this equation must hold true whatever be the value of y , take $y = x$, and it will become

$$\frac{m}{n} \times 1 + x^{\frac{m}{n}-1} = a + 2bx + 3cx^2 + 4cx^3 \&c.$$

Consequently, multiplying by $1 + x$, we have

$$\frac{m}{n} \times 1 + x^{\frac{m}{n}}, \text{ or its equal by the assumption,}$$

$$\text{viz. } \frac{m}{n} + \frac{m}{n}ax + \frac{m}{n}bx^2 + \frac{m}{n}cx^3 \&c.$$

$$= a + \frac{2b}{a} \} x + \frac{3c}{2b} \} x^2 + \frac{4d}{3c} \} x^3 \&c.$$

Then, by comparing the homologous terms, the value of the coefficients $a, b, c, \&c.$ are deduced for as many terms as you compare.

And a large account is given of this investigation by the learned Dr. Hales, in his *Analysis Equationum*, lately published at Dublin.

Mr. Landen then contrasts this investigation with that by the method of fluxions, which is as follows. Assume as before;

$$\frac{m}{1+x^n} = 1 + ax + bx^2 + cx^3 + dx^4 \&c.$$

Take the fluxion of each side, and we have

$$\frac{m}{n} \times \frac{m}{1+x^n}^{-1} \times \dot{x} = a\dot{x} + 2bx\dot{x} + 3cx^2\dot{x} \&c.$$

Divide by $\dot{x}$, or take it = 1, so shall

$$\frac{m}{n} \times \frac{m}{1+x^n}^{-1} = a + 2bx + 3cx^2 + 4dx^3 \&c.$$

Then multiply by $1+x$, and so on as above in the other way.

16. Besides the above, which are the principal demonstrations and investigations that have been given of this important theorem, I have been shewn an ingenious attempt of Mr. Baron Maseres, to demonstrate this theorem in the case of roots or fractional exponents, by the help of De Moivre's multinomial theorem. But, not being quite satisfied with his own demonstration, as not expressing the law of continuation of the terms which are not actually set down, he was pleased to urge me to attempt a more complete and satisfactory demonstration of the general case of roots, or fractional exponents. And he farther proposed it in this form, namely, that if Q be the coefficient of one of the terms of the series which is equal to $\frac{1}{1+x^n}$, and P the coefficient of the next preceding term, and R the coefficient of the next following term; then, if Q be $= \frac{a}{b} \times P$, to prove that R will be $= \frac{a-n}{b+n} \times Q$. This he observed would be quite perfect and satisfactory,

as

as it would include all the terms of the series, as well those that are omitted, as those that are actually set down. And I was, in my demonstration, to suppose, if I pleased, the truth of the binomial and multinomial theorems for integral powers, as truths that had been previously and perfectly proved.

In consequence I sent him soon after the substance of the following demonstration; with which he was quite satisfied, and which I now proceed to explain at large.

17. Now the binomial integral is $\sqrt[n]{1+x}$

$$= 1 + \overset{a}{\frac{n}{1}}x + \overset{b}{\frac{n}{1} \cdot \frac{n-1}{2}}x^2 + \overset{c}{\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}}x^3 + \overset{d}{\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}}x^4 + \&c.$$

or

$$= 1 + \frac{n}{1}x + \frac{n-1}{2}ax^2 + \frac{n-2}{3}bx^3 + \frac{n-3}{4}cx^4 + \&c.$$

where $a, b, c, \&c.$ denote the whole coefficients of the 2d, 3d, 4th, &c. terms, over which they are placed; and in which the law is this, namely, if P, Q, R , be the coefficients of any three terms in succession, and if $\frac{g}{b}P = Q$, then is $\frac{g-1}{b+1}Q = R$; as is evident; and which, it is granted, has been proved.

18. And the binomial fractional is $\sqrt[n]{1+x}$

$$= 1 + \frac{1}{n}x + \frac{1}{n} \cdot \frac{1-n}{2n}x^2 + \frac{1}{n} \cdot \frac{1-n}{2n} \cdot \frac{1-2n}{3n}x^3 + \frac{1}{n} \cdot \frac{1-n}{2n} \cdot \frac{1-2n}{3n} \cdot \frac{1-3n}{4n}x^4 + \&c.$$

or

$$= 1 + \frac{1}{n}x + \frac{1-n}{2n}ax^2 + \frac{1-2n}{3n}bx^3 + \frac{1-3n}{4n}cx^4 + \&c.$$

in which the law is this, namely, if P, Q, R be the coefficients of three terms in succession; and if $\frac{g}{b}P = Q$, then is $\frac{g-n}{b+n}Q = R$. Which is the property to be proved.

19. Again, the multinomial integral is

$$\begin{aligned}
 & \frac{1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots}{1 + Ax + Bx^2 + Cx^3 + Dx^4 + \dots}^n = 1 \\
 & \quad (a) \qquad \qquad \qquad + \frac{n}{1} A x \\
 & \quad (b) \qquad \qquad \qquad + \frac{n}{1} \cdot \frac{n-1}{2} A^2 x^2 \\
 & \qquad \qquad \qquad + \frac{n}{1} B \\
 & \qquad \qquad \qquad + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} A^3 x^3 \\
 & \quad (c) \qquad \qquad \qquad + \frac{n}{1} \cdot \frac{n-1}{1} AB \\
 & \qquad \qquad \qquad + \frac{n}{1} C \\
 & \qquad \qquad \qquad + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} A^4 x^4 \\
 & \qquad \qquad \qquad + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{1} A^2 B \\
 & \quad (d) \qquad \qquad \qquad + \frac{n}{1} \cdot \frac{n-1}{1} AC \\
 & \qquad \qquad \qquad + \frac{n}{1} \cdot \frac{n-1}{2} B^2 \\
 & \qquad \qquad \qquad + \frac{n}{1} D \\
 & \qquad \qquad \qquad + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5} A^5 x^5 \\
 & \qquad \qquad \qquad + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-3}{3} \cdot \frac{n-3}{1} A^3 B \\
 & \qquad \qquad \qquad + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{1} A^2 C \\
 & \quad (e) \qquad \qquad \qquad + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{2} AB^2 \\
 & \qquad \qquad \qquad + \frac{n}{1} \cdot \frac{n-1}{2} AD \\
 & \qquad \qquad \qquad + \frac{n}{1} \cdot \frac{n-1}{1} BC \\
 & \qquad \qquad \qquad + \frac{n}{1} E
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5} \cdot \frac{n-5}{6} A^6 x^6 \\
 & + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{1} A^4 B \\
 & + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{1} A^3 C \\
 & + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{2} \cdot \frac{n-3}{1} A^2 B^2 \\
 & + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{1} A^2 D \\
 & + \frac{n}{1} \cdot \frac{n-1}{1} \cdot \frac{n-2}{1} ABC \\
 & + \frac{n}{1} \cdot \frac{n-1}{1} AE \\
 & + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} B^3 \\
 & + \frac{n}{1} \cdot \frac{n-1}{1} BD \\
 & + \frac{n}{1} \cdot \frac{n-1}{2} C^2 \\
 & + \frac{n}{1} F
 \end{aligned}$$

&c.

Or, if we put $a, b, c, d, \&c.$ for the coefficients of the 2d, 3d, 4th, 5th, &c. terms, the last series, by substitution, will be transformed into this form,

$$\begin{aligned}
 & (1 + Ax + Bx^2 + Cx^3 + \&c.)^n = - - - - - I \\
 (a) & \qquad \qquad \qquad + \frac{nA}{1} x \\
 (b) & \qquad \qquad \qquad + \frac{2nB + \overline{n-1} \cdot Aa}{2} x^2 \\
 (c) & \qquad \qquad \qquad + \frac{3nC + \overline{2n-1} \cdot Ba + \overline{n-2} \cdot Ab}{3} x^3 \\
 (d) & \qquad \qquad \qquad + \frac{4nD + \overline{3n-1} \cdot Ca + \overline{2n-2} \cdot Bb + \overline{n-3} \cdot Ac}{4} x^4 \\
 (e) & \qquad \qquad \qquad + \frac{5nE + \overline{4n-1} \cdot Da + \overline{3n-2} \cdot Cb + \overline{2n-3} \cdot Bc + \overline{n-4} \cdot Ad}{5} x^5 \\
 (f) & + \frac{6nF + \overline{5n-1} \cdot Ea + \overline{4n-2} \cdot Db + \overline{3n-3} \cdot Cc + \overline{2n-4} \cdot Bd + \overline{n-5} \cdot Ae}{6} x^6 \\
 & \qquad \qquad \qquad \&c.
 \end{aligned}$$

20. Now, to find the series in Art. 18, assume the proposed binomial equal to a series with indeterminate coefficients, as

$$1 + x^n = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$$

Then raise each side to the n power, so shall

$$1 + x = (1 + Ax + Bx^2 + Cx^3 + \&c.)^n.$$

But it is granted that the multinomial raised to any integral power is proved, and known to be, as in the last Art.

$$\begin{aligned} 1 + x &= (1 + Ax + Bx^2 + Cx^3 + \&c.)^n = \\ &1 + \overbrace{\frac{nA}{1}x}^a + \overbrace{\frac{2nB + n-1.Aa}{2}x^2}^b + \overbrace{\frac{3nC + 2n-1.Ba + n-2.Ab}{3}x^3}^c \\ &+ \overbrace{\frac{4nD + 3n-1.Ca + 2n-2.Bb + n-3.Ac}{4}x^4}^d + \&c. \end{aligned}$$

It follows then, that if this last series be equal to $1 + x$, by equating the homologous coefficients, all the terms after the second must vanish, or all the coefficients $b, c, d, \&c.$ after the second term, must be each $= 0$. Writing therefore, in this series, 0 for each of the letters $b, c, d, \&c.$ it will become of this more simple form,

$$\begin{aligned} 1 + x &= \\ &1 + \overbrace{\frac{nA}{1}x}^a + \overbrace{\frac{2nB + n-1.Aa}{2}x^2}^{b=0} + \overbrace{\frac{3nC + 2n-1.Ba}{3}x^3}^{c=0} + \overbrace{\frac{4nD + 3n-1.Ca}{4}x^4}^{d=0} + \&c. \end{aligned}$$

Put now each of the coefficients, after the second term, $= 0$, and we shall have these equations

$$\begin{aligned} 2nB + n-1.Aa &= 0 \\ 3nC + 2n-1.Ba &= 0 \\ 4nD + 3n-1.Ca &= 0 \\ 5nE + 4n-1.Da &= 0 \\ &\&c. \end{aligned}$$

The

The resolution of which equations gives the following values of the assumed indeterminate coefficients, namely,

$$B = \frac{1-n}{2n} Aa, \quad C = \frac{1-2n}{3n} Ba, \quad D = \frac{1-3n}{4n} Ca, \quad E = \frac{1-4n}{5n} Da, \quad \&c.$$

which coefficients are according to the law proposed, namely, when

$$\frac{g}{b} P = Q, \text{ then } \frac{g-n}{b+n} Q = R. \quad \mathcal{Q}. E. D.$$

21. Also, by equating the second coefficients, namely, $1 = a = nA$, we find $A = \frac{1}{n}$. This being written for A in the above values of $B, C, D, \&c.$ will give the proper series for the binomial in question, namely

$$\begin{aligned} \sqrt[n]{1+x} &= 1 + Ax + Bx^2 + Cx^3 + \&c. \\ &= 1 + \frac{1}{n}x + \frac{1-n}{2n}ax^2 + \frac{1-2n}{3n}bx^3 + \&c. \\ &= 1 + \frac{1}{n}x + \frac{1}{n} \cdot \frac{1-n}{2n}x^2 + \frac{1}{n} \cdot \frac{1-n}{2n} \cdot \frac{1-2n}{3n}x^3 + \&c. \end{aligned}$$

Of the FORM of the ASSUMED SERIES.

22. In the demonstrations or investigations of the truth of the binomial theorem, the but or object has always been the law of the coefficients of the terms: the form of the series, as to the powers of x , having never been disputed, but taken for granted, either as incapable of receiving a demonstration, or as too evident to need one. But since the demonstration of the law of the coefficients has been accomplished, in which the main, if not the only, difficulty was supposed to consist, we have extended our researches still farther, and have even doubted or queried the very *form* of the terms themselves, namely, $1 + Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$ increasing by the regular integral series of the powers of x , as assumed to denote the quantity $\sqrt[n]{1+x}$, or the n root of $1+x$. And in consequence of these scruples, I have been required, by a learned friend, to vindicate the propriety

propriety of that assumption. Which I think is effectually done as follows.

23. To prove then, that any root of the binomial $1 + x$ can be represented by a series of this form $1 + x + x^2 + x^3 + x^4 \&c.$ where the coefficients are omitted, our attention being now employed only on the powers of x ; let the series representing the value of $\sqrt[n]{1 + x^n}$ be $1 + A + B + C + D + \&c.$ where $A, B, C, \&c.$ now represent the whole of the 2d, 3d, 4th, $\&c.$ terms, both their coefficients and the powers of x , whatever they may be, only increasing from the left to the greater, because they increase in the terms $1 + x$ of the given binomial itself; and in which the first term must evidently be 1, the same as in the given binomial.

Raise now $\sqrt[n]{1 + x^n}$ and its equivalent series $1 + A + B + C + \&c.$ both to the n power by the multinomial theorem, and we shall have, as before,

$$\begin{aligned}
 1 + x = & \\
 1 + \frac{n}{1}A + \frac{n}{1} \cdot \frac{n-1}{2}A^2 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}A^3 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}A^4 \&c. \\
 & \frac{n}{1}B \qquad \frac{n}{1} \cdot \frac{n-1}{1}AB \qquad \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{1}A^2B \\
 & \qquad \qquad \frac{n}{1}C \qquad \qquad \frac{n}{1} \cdot \frac{n-1}{1}AC \\
 & \qquad \qquad \qquad \frac{n}{1} \cdot \frac{n-1}{2}B^2 \\
 & \qquad \qquad \qquad \qquad \frac{n}{1}D
 \end{aligned}$$

Then equate the corresponding terms, and we have the first term $1 = 1$,

Again, the second term of the series $\frac{n}{1}A$, must be equal to the second term x of the binomial. For none of the other terms of the series are equipollent, or contain the same power of x , with the term $\frac{n}{1}A$. Not any of the terms $A^2, A^3, A^4, \&c.$ for they are double, triple, quadruple, $\&c.$ in power to A . Nor yet any of the terms containing $B, C, D, \&c.$ because, by the supposition, they contain all different and increasing

increasing powers. It follows therefore, that $\frac{n}{1}A$ makes up the whole value of the second term x of the given binomial. Consequently the second term A of the assumed series, contains only the first power of x ; and the whole value of that term A is $= \frac{1}{n}x$.

But all the other equipollent terms of the expanded series must be equal to nothing, which is the general value of the terms, after the second, of the given quantity $1 + x$ or $1 + x + 0 + 0 + 0 + \&c$. Our business is therefore to find the several orders of equipollent terms of the expanded series. And these I say will be as I have arranged them above, in which B is equipollent with A^2 , C with A^3 , D with A^4 , and so on.

Now that B is equipollent with A^2 , is thus proved. The value of the third term is 0 . But $\frac{n}{1} \cdot \frac{n-1}{2} A^2$ is a part of the third term. And it is only a part of that term: otherwise $\frac{n}{1} \cdot \frac{n-1}{2}$ would be $= 0$, which it is evident cannot happen in every value of n , as it ought; for indeed it happens only when n is $= 1$. Some other quantity then must be equipollent with $\frac{n}{1} \cdot \frac{n-1}{2} A^2$, and must be joined with it, to make up the whole third term equal to 0 . Now that supplemental quantity can be no other than $\frac{n}{1} B$: for all the other following terms are evidently plupollent than B . It follows therefore, that B is equipollent with A^2 , and contains the second power of x ; or that $\frac{n}{1} \cdot \frac{n-1}{2} A^2 + \frac{n}{1} B = 0$, and consequently

$$\frac{n-1}{2} A^2 + B = 0, \text{ or } B = \frac{1-n}{2} A^2 = \frac{1-n}{2n} Ax = \frac{1}{n} \cdot \frac{1-n}{2n} x^2.$$

Again, the fourth term must be $= 0$. But the quantities $\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} A^3 + \frac{n}{1} \cdot \frac{n-1}{2} AB$ are equipollent, and make up part of that fourth term. They are equipollent, or A^3 equipollent with AB , because A^2 and B are equipollent. And they do not

constitute the whole of that term; for if they did, then would $\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} A^3 + \frac{n}{1} \cdot \frac{n-1}{2} AB$ be $= 0$ in all values of n , or $\frac{n-2}{3} A^2 + B = 0$: but it has been just shewn above, that $\frac{n-1}{2} A^2 + B = 0$; it would therefore follow that $\frac{n-2}{3}$ would be $= \frac{n-1}{2}$, a circumstance which can only happen where $n = -1$, instead of taking place for every value of n . Some other quantity must therefore be joined with these to make up the whole of the fourth term. And this supplemental quantity can be no other than $\frac{n}{1} c$, because all the other following quantities are evidently plupollent than A^3 or AB . It follows therefore, that c is equipollent with A^3 , and therefore contains the 3d power of x . And the whole value of c is $= \frac{1-n}{2} \cdot \frac{n-2}{3} A^3 + \frac{1-n}{1} AB = \frac{1-2n}{3} AB = \frac{1-2n}{3n} Bx = \frac{1}{n} \cdot \frac{1-n}{2n} \cdot \frac{1-2n}{3n} x^3$.

And the process is the same for all the other following terms. Thus, then, we have proved the law of the whole series, both with respect to the coefficients of its terms, and to the powers of the letter x .



VII.

Of the Common Sections of the Sphere and Cone. Together with the Demonstration of some other New Properties of the Sphere, which are similar to certain Known Properties of the Circle.

THE study of the mathematical sciences is useful and profitable, not only on account of the benefit derivable from them to the affairs of mankind in general; but are most eminently so, for the pleasure and delight the human mind feels in the discovery and contemplation of the endless number of truths that are continually presenting themselves to our view. These meditations are of a sublimity far above all others, whether they be purely intellectual, or whether they respect the nature and properties of material objects: they methodise, strengthen, and extend the reasoning faculties in the most eminent degree, and so fit the mind the better for understanding and improving every other science; but, above all, they furnish us with the purest and most permanent delight, from the contemplation of truths peculiarly certain and immutable, and from the beautiful analogy which reigns through all the objects of similar inquiry. In the mathematical sciences, the discovery, often accidental, of a plain and simple property, is but the harbinger of a thousand others of the most sublime and beautiful nature, to which we are gradually led, delighted, from the more simple to the more compound and general, till the mind becomes quite enraptured at the full blaze of light bursting upon it from all directions.

Of these very pleasing subjects, the striking analogy that prevails among the properties of geometrical figures, or figured extension, is not one of the least. Here we often find that a plain and obvious property of one of the simplest figures, leads us to, and forms only a particular case of, a property in some other figure, less simple; afterwards this again turns out to be no more than a particular case of another still more general; and so on, till at last we often trace the tendency to end in a general property of all figures whatever.

The few properties which make a part of this paper, constitute a small specimen of the analogy, and even identity, of some of the more remarkable properties of the circle, with those of the sphere. To which are added some properties of the lines of section, and of contact, between the sphere and cone. Both which may be farther extended as occasions may offer: like as all of these properties have occurred from the circumstance, mentioned near the end of the paper, of considering the inner surface of a hollow spherical vessel, as viewed by an eye, or as illuminated by rays, from a given point.

PROPOSITION I.

All the tangents are equal, which are drawn, from a given point without a sphere, to the surface of the sphere quite around.

DEMONS. For, let PT be any tangent from the given point P ; and draw PC to the center C , and join TC . Also let CTA be a great circle of the sphere in the plane of the triangle TPC . Then, CP and CT , as well as the angle T , which is right (Eucl. iii. 18), being constant, in every position of the tangent, or of the point of contact T ; the square of PT will be every where equal to the difference of the squares of the constant lines CP , CT , and therefore constant; and consequently the line or tangent PT itself of a constant length, in every position, quite round the surface of the sphere.



If

PROP. II.

If a tangent be drawn to a sphere, and a radius be drawn from the center to the point of contact, it will be perpendicular to the tangent; and a perpendicular to the tangent will pass through the center.

DEMONS. For, let PT be the tangent, TC the radius, and CTA a great circle of the sphere in the plane of the triangle TPC , as in the foregoing proposition. Then, PT touching the circle in the point T , the radius TC is perpendicular to the tangent PT by Eucl. iii. 18, 19.

PROP. III.

If any line or chord be drawn in a sphere, its extremes terminating in the circumference; then a perpendicular drawn to it from the center, will bisect it: and if the line drawn from the center, bisect it, it is perpendicular to it.

DEMONS. For, a plane may pass through the given line and the center of the sphere; and the section of that plane with the sphere, will be a great circle (Theodos. i. 1), of which the given line will be a chord. Therefore (Eucl. iii. 3) the perpendicular bisects the chord, and the bisecting line is perpendicular.

COROL. A line drawn from the center of the sphere, to the center of any lesser circle, or circular section, is perpendicular to the plane of that circle. For, by the proposition, it is perpendicular to all the diameters of that circle.

PROP. IV.

If from a given point, a right line be drawn in any position through a sphere, cutting its surface always in two points; the rectangle contained under the whole line and the external part, that is the rectangle contained by the two distances between the given point, and the two points where the line meets the surface of the sphere, will always be of
the

the same constant magnitude, namely, equal to the square of the tangent drawn from the same given point.

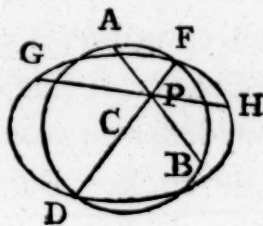
DEMONS. Let P be the given point, and AB the two points in which the line PAB meets the surface of the sphere: through PAB and the center let a plane cut the sphere in the great circle TAB , to which draw the tangent PT . Then the rectangle $PA \cdot PB$ is equal to the square of PT (Eucl. iii. 36); but PT , and consequently its square, is constant by Prop. 1; therefore the rectangle $PA \cdot PB$, which is always equal to this square, is every where of the same constant magnitude.



PROP. V.

If any two lines intersect each other within a sphere, and be terminated at the surface on both sides; the rectangle of the parts of the one line, will be equal to the rectangle of the parts of the other. And, universally, the rectangles of the two parts of all lines passing through the point of intersection, are all of the same magnitude.

DEMONS. Through any one of the lines, as AB , conceive a plane to be drawn through the center C of the sphere, cutting the sphere in the great circle ABD ; and draw its diameter $DCPF$ through the point of intersection P of all the lines. Then the rectangle $AP \cdot PB$ is equal to the rectangle $DP \cdot PF$ (Eucl. iii. 35).



Again, through any other of the intersecting lines GH , and the center, conceive another plane to pass, cutting the sphere in another great circle $DGFH$. Then, because the points C and P are in this latter plane, the line CP , and consequently the whole diameter $DCPF$, is in the same plane; and therefore it is a diameter of the circle $DGFH$, of which GPH is a chord. Therefore, again, the rectangle $GP \cdot PH$ is equal to the rectangle $DP \cdot PF$ (Eucl. iii. 35)

Consequently

Conſequently all the rectangles $AP.PB$, $GP.PH$, &c. are equal, being each equal to the conſtant rectangle $DP.PF$.

COROL. The great circles paſſing through all the lines or chords which interſect in the point P , will all interſect in the common diameter DPF .

PROP. VI.

If a ſphere be placed within a cone, ſo as to touch it in two points; then ſhall the outside of the ſphere, and the inſide of the cone, mutually touch quite around, and the line of contact will be a circle.

DEMONS. Let V be the vertex of the cone, C the center of the ſphere, T one of the two points of contact, and TV a ſide of the cone. Draw CT , CV . Then TVC is a triangle right-angled at T (Prop. 2). In like manner, t being another point of contact, and Ct being drawn, the triangle tVC will be right-angled at t . Theſe two triangles then, TVC , tVC , having the two ſides CT , CV , equal to the two Ct , tV (Prop. 1), and the included angle T equal to the included angle t , will be equal in all reſpects (Eucl. i. 4), and conſequently have the angle TVC equal to the angle tVC .



Again, let fall the perpendiculars TP , tP . Then the two triangles TVP , tVP , having the two angles TVP and TPV equal to the two tVP and tPV , and the ſide TV equal to the ſide tV (Prop. 1), will be equal in all reſpects (Eucl. i. 26); conſequently TP is equal to tP , and VP equal to VP . Hence PT , Pt are radii of a little circle of the ſphere, whoſe plane is perpendicular to the line CV , and its circumference every where equidistant from the point C or V . This circle is therefore a circular ſection both of the ſphere and of the cone, and is therefore the line of their mutual contact. Alſo CV is the axis of the cone.

COROL.

COROL. 1. The axis of a cone, when produced, passes through the center of the inscribed sphere.

COROL. 2. Hence also, every cone circumscribing a sphere, so that their surfaces touch quite around, is a right cone; nor can any scalene or oblique cone touch a sphere in that manner.

PROP. VII.

The two common sections of the surfaces of a sphere and a right cone, are the circumferences of circles if the axis of the cone pass through the center of the sphere.

DEMONS. Let V be the vertex of the cone, C the center of the sphere, and S one point of the less or nearer section; draw the lines CS , CV . Then, in the triangle CSV , the two sides CS , CV , and the included angle SCV , are constant for all positions of the side VS ; and therefore the side VS is of a constant length for all positions, and is consequently the side of a right cone having a circular base; therefore the locus of all the points S , is the circumference of a circle perpendicular to the axis CV , that is, the common section of the surfaces of the sphere and cone, is that circumference.



In the same manner it is proved that, if A be any point in the farther or greater section, and CA be drawn; then VA is constant for all positions, and therefore, as before, is the side of a cone cut off by a circular section whose plane is perpendicular to the axis.

And these circles, being both perpendicular to the axis, are parallel to each other. Or, they are parallel because they are both circular sections of the cone.

COROL. 1. Hence $SA = sa$, because $VA = Va$, and $VS = Vs$.

COROL. 2. All the intercepted equal parts SA , sa , &c. are equally distant from the center. For, all the sides of the triangle SCA

are

are constant, and therefore the perpendicular CP is constant also. And thus all the equal right lines or chords in a sphere, are equally distant from the center.

COROL. 3. The sections are not circles, and therefore not in planes, if the axis pass not through the center. For then some of the points of section are farther from the vertex than others.

PROP. VIII.

Of the two common sections of a sphere and an oblique cone, if the one be a circle, the other will be a circle also.

DEMONS. Let SAas and ASVa be sections of the sphere and cone, made by a common plane passing through the axes of the cone and the sphere; also Ss, Aa the diameters of the two sections. Now, by the supposition, one of these, as Aa, is the diameter of a circle. But the angle VSs = the angle VaA (Eucl. i. 13, and iii. 22), therefore Ss cuts the cone in sub-contrary position to Aa; and consequently if a plane pass through Ss, and perpendicular to the plane AVa, its section with the oblique cone will be a circle, whose diameter is the line Ss (Apol. i. 5). But the section of the same plane and the sphere, is also a circle whose diameter is the same line Ss (Theod. i. 1). Consequently the circumference of the same circle, whose diameter is Ss, is in the surface both of the cone and sphere; and therefore that circle is the common section of the cone and sphere.



In like manner, if the one section be a circle whose diameter is Sa, the other section will be a circle whose diameter is sA.

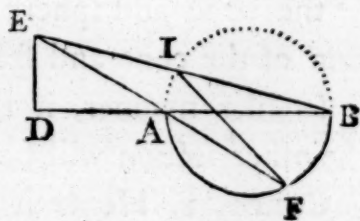
COROL. 1. Hence if the one section be not a circle, neither of them is a circle; and consequently they are not in planes; for the section of a sphere by a plane, is a circle.

COROL. 2. When the sections of a sphere and oblique cone are circles, the axis of the cone does not pass through the center of the

sphere, (except when one of the sections is a great circle, or passes through the center). For the axis passes through the center of the base, but not perpendicularly; whereas a line drawn from the center of the sphere to the center of the base, is perpendicular to the base, by Cor. to Prop. 3.

COROL. 3. Hence, if the inside of a bowl, which is a hemisphere, or any segment of the sphere, be viewed by an eye not situated in the axis produced, which is perpendicular to the section or brim; the lower, or extreme part of the internal surface which is visible, will be bounded by a circle of the sphere; and the part of the surface seen by the eye, will be included between the said circle, and the border or brim, which it intersects in two points. For the eye is in the place of the vertex of the cone; and the rays from the eye to the brim of the bowl, and thence continued from the nearer part of the brim, to the opposite internal surface, form the sides of the cone; which, by the proposition, will form a circular arc on the said internal surface; because the brim, which is the one section, is a circle.

And hence, the place of the eye being given, the quantity of internal surface that can be seen, may be easily determined. For the distance and height of the eye, with respect to the brim, will give the greatest distance of the section below the brim, together with its magnitude and inclination to the plane of the brim; which being known, common mensuration furnishes us with the measure of the surface included between them. Thus, if AB be the diameter in the vertical plane passing through the eye at E, also AFB the section of the bowl by the same plane, and AIB the supplement of that arc. Draw EAF, EIB, cutting this vertical circle in F and I; and join IF. Then shall IF be the diameter of the section or extremity of the visible surface, and BF its greatest distance below the brim, an arc which measures an angle double the angle at A.



COROL.

COROL. 4. Hence also, and from Proposition 4, it follows, that if through every point in the circumference of a circle, lines be drawn to a given point E out of the plane of the circle, so that the rectangle contained under the parts between the point E and the circle, and between the same point E and some other point F, may always be of a certain given magnitude; then the locus of all the points F will also be a circle, cutting the former circle in the two points where the lines drawn from the given point E, to the several points in the circumference of the first circle, change from the convex to the concave side of the circumference. And the constant quantity, to which the rectangle of the parts is always equal, is equal to the square of the line drawn from the given point E to either of the said two points of intersection.

And thus the loci of the extremes of all such lines, are circles.

PROP. IX. *Prob.*

To place a given sphere, and a given oblique cone, in such positions, that their mutual sections shall be circles.

Let V be the vertex, VB the least side, and VD the greatest side of the cone. In the plane of the triangle VBD it is evident will be found the center of the sphere. Parallel to BD draw Aa the diameter of a circular section of the cone, so that it be not greater than the diameter of the sphere. Bisect Aa with the perpendicular EC; with the center A and radius of the sphere, cut EC in C, which will be the center of the sphere; from which therefore describe a great circle of it cutting the sides of the cone in the points S, s, A, a: so shall Ss and Aa be the diameters of circular sections which are common to both the sphere and cone.



NOTE. The substance of the above propositions was drawn up several years ago. And Mr. Bonnycastle and Mr. George Sanderfon have this day shewn me the solution of a question in the London Magazine for April 1777, in which a similar section of a sphere with a cone, is proved to be a circle, and which I had never seen before. Nor do I know of any other writings on the same subject.

July 29, 1785.



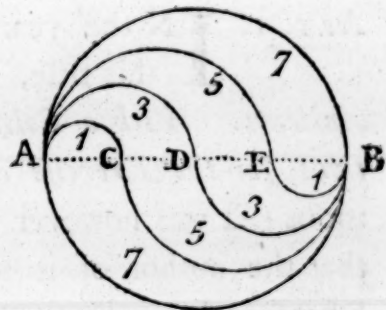
Of the Geometrical Division of Circles and Ellipses into any Number of Parts, and in any proposed Ratios.

ART. I. **I**N the year 1774 was published a pamphlet in octavo, with this title, *A Dissertation on the Geometrical Analysis of the Antients. With a Collection of Theorems and Problems, without Solutions, for the Exercise of Young Students.* This pamphlet was anonymous; it was however well known to myself and several other persons, that the author of it was the late Mr. John Lawson, B. D. rector of Swanscombe in Kent, an ingenious and learned geometrician, and, what is still more estimable, a most worthy and good man; one in whose heart was found no guile, and whose pure integrity, joined to the most amiable simplicity of manners, and sweetness of temper, gained him the affection and respect of all who had the happiness to be acquainted with him. His collection of problems in that pamphlet concluded with this singular one, "To divide a circle into any number of parts, which shall be as well equal in area as in circumference.—N. B. *This may seem a paradox, however it may be effected in a manner strictly geometrical.*" The solution of this seeming paradox he reserved to himself, as far as I know. I fell upon the discovery however soon after; and other persons might do the same. My resolution of it was published in an account which I gave of the pamphlet in the Critical Review for 1775, vol. xl. and which the author informed me was on the

the same principle as his own. This account is in page 21 of that volume, and in the following words:

2. "We have no doubt but that our mathematical readers will agree with us in allowing the truth of the author's remark concerning the seeming paradox of this problem; because there is no geometrical method of dividing the circumference of a circle into any proposed number of parts taken at pleasure, and it does not readily appear that there can be any other method of resolving the problem, than by drawing radii to the points of equal division in the circumference. However another method there is, and that strictly geometrical, which is as follows.

"Divide the diameter AB of the given circle into as many equal parts as the circle itself is to be divided into, at the points C, D, E, &c. Then on the diameters AC, AD, AE, &c. as also on BE, BD, BC, &c. describe semicircles, as in the annexed figure: and they will divide the whole circle as required.



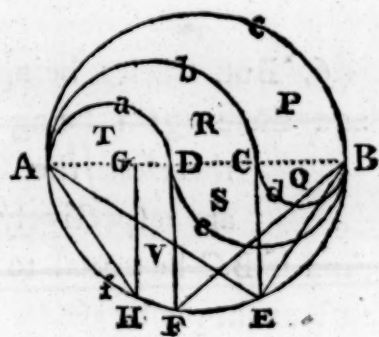
"For, the several diameters being in arithmetical progression, of which the common difference is equal to the least of them, and the diameters of circles being as their circumferences, these will also be in arithmetical progression. But, in such a progression, the sum of the extremes is equal to the sum of each two terms equally distant from them; therefore the sum of the circumferences on AC and CB, is equal to the sum of those on AD and DB, and of those on AE and EB, &c. and each sum equal to the semi-circumference of the given circle on the diameter AB. Therefore all the parts have equal perimeters, and each is equal to the circumference of the proposed circle. Which satisfies one of the conditions in the problem.

"Again

“ Again, the same diameters being as the numbers 1, 2, 3, 4, &c. and the areas of circles being as the squares of their diameters, the semicircles will be as the numbers 1, 4, 9, 16, &c. and consequently the differences between all the adjacent semicircles are as the terms of the arithmetical progression 1, 3, 5, 7, &c. and here again the sums of the extremes, and of every two equidistant means, make up the several equal parts of the circle. Which is the other condition.”

3. But this subject admits of a more geometrical form, and is capable of being rendered very general and extensive, and is moreover very fruitful in curious consequences. For first, in whatever ratio the whole diameter is divided, whether into equal or unequal parts, and whatever be the number of the parts, the perimeters of the spaces will still be equal.

For since circumferences of circles are always as their diameters, and because AB and $AD + DB$ and $AC + CB$ are all equal, therefore the semi-circumferences c and $b + d$ and $a + e$ are all equal, and constant, whatever be the ratio of the parts AD , DC , CB , of the diameter.



We shall presently find too that the spaces TV , RS , and PQ , will be universally as the same parts AD , DC , CB , of the diameter.

4. The semicircles having been described as before mentioned, erect CE perpendicular to AB , and join BE . Then I say, the circle on the diameter BE , will be equal to the space PQ . For, join AE .

Now

Now the space $P = \text{semicircle on } AB - \text{semicircle on } AC$;
 but the femicir. on $AB = \text{femicir. on } AE + \text{femicir. on } BE$,
 and the femicir. on $AC = \text{femicir. on } AE - \text{femicir. on } CE$,
 theref. femic. $AB - \text{femic. } AC = \text{femic. } BE + \text{femicir. } CE$,
 that is the space P is $= \text{femic. } BE + \text{femicir. } CE$;
 to each of these add the space Q , or the semicircle on BC ,
 then $P + Q = \text{femic. } BE + \text{femic. } CE + \text{femic. } BC$,
 that is $P + Q = \text{double the femic. } BE$, or $= \text{the whole circle on } BE$.

5. In like manner, the two spaces PQ and RS together, or the whole space $PQRS$, is equal to the circle on the diameter BF . And therefore the space RS alone, is equal to the difference, or the circle on BF *minus* the circle on BE .

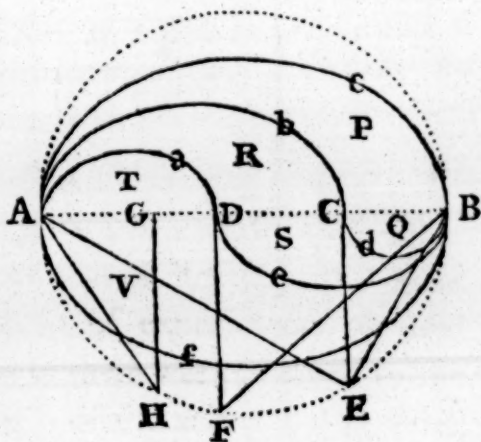
6. But, circles being as the squares of their diameters, BE^2 , BF^2 , and these again being as the parts or lines BC , BD ,
 therefore the spaces PQ , $PQRS$, RS , TV ,
 are respectively as the lines BC , BD , CD , AD ,
 And if BC be equal to CD , then will PQ be equal to RS , as in the first or simplest case.

7. Hence, to find a circle equal to the space RS , where the points D and C are taken at random: From either end of the diameter, as A , take AG equal to DC , erect GH perpendicular to AB , and join AH ; then the circle on AH will be equal to the space RS . For, the space PQ : the space $RS :: BC : CD$ or AG , that is as $BE^2 : AH^2$ the squares of the diameters, or as the circle on BE to the circle on AH ; but the circle on BE is equal to the space PQ , and therefore the circle on AH is equal to the space RS .

8. Hence, to divide a circle in this manner, into any number of parts, that shall be in any ratios to one another: Divide the diameter
 into

into as many parts, at the points D, C, &c. and in the same ratios as those proposed; then on the several distances of these points from the two ends A and B, as diameters, describe the alternate semicircles on the different sides of the whole diameter AB: and they will divide the whole circle in the manner proposed. That is, the spaces TV, RS, PQ, will be as the lines AD, DC, CB.

9. But these properties are not confined to the circle alone, but are to be found also in the ellipse, as the genus of which the circle is only a species. For if the annexed figure be an ellipse described on the axis AB, the area of which is, in like manner, divided by similar semiellipses, described on AD, AC, BC, BD, as axes, all the semiperimeters f , a , e , bd , c , will be equal to one another, for the same reason as before in Art. 3, namely, because the peripheries of ellipses are as their diameters. And the same property would still hold good, if AB were any other diameter of the ellipse, instead of the axis; describing upon the parts of it semiellipses which shall be similar to those into which the diameter AB divides the given ellipse.



10. And, if a circle be described about the ellipse, on the diameter AB, and lines be drawn similar to those in the second figure; then, by a process the very same as in Art. 4, *et seq.* substituting only semiellipse for semicircle, it is found that the space

PQ is equal to the similar ellipse on the diameter BE,

PQRS is equal to the similar ellipse on the diameter BF,

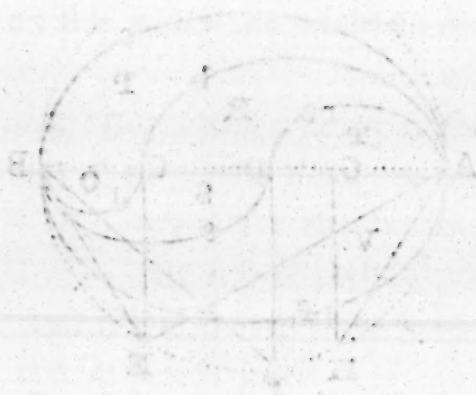
RS is equal to the similar ellipse on the diameter AH,

or to the difference of the ellipses on BF and BE;

also the elliptic spaces - - PQ, PQRS, RS, TV,

are respectively as the lines - BC, BD, DC, AD,

the same ratio as the circular spaces. And hence an ellipse is divided into any number of parts, in any assigned ratios, in the same manner as the circle is divided, namely, dividing the axis, or any diameter in the same manner, and on the parts describing similar semi-ellipses.





New Experiments in Artillery; for determining the Force of fired Gunpowder, the Initial Velocity of Cannon Balls, the Ranges of Pieces of Cannon at different Elevations, the Resistance of the Air to Projectiles, the Effect of different Lengths of Cannon, and of different Quantities of Powder, &c. &c.

Sect. I. **A**T Woolwich in the year 1775, in conjunction with some able officers of the Royal Regiment of Artillery, and other ingenious gentlemen, I first instituted a course of experiments on fired gunpowder and cannon balls. My account of them was presented to the Royal Society, who honoured it with the gift of the annual gold medal, and printed it in the Philosophical Transactions for the year 1778. The object of those experiments, was the determination of the actual velocities with which balls are impelled from given pieces of cannon, when fired with given charges of powder. They were made according to the method invented by the very ingenious Mr. Robins, and described in his treatise on the new principles of gunnery, of which an account was printed in the Philosophical Transactions for the year 1743. Before the discoveries and inventions of that gentleman, very little progress had been made in the true theory of military projectiles. His book however contained such important discoveries, that it was soon translated into several of the languages on the continent, and the late famous Mr. L. Euler honoured it with a very learned and extensive commentary, in his translation of it into the German language. That

part of Mr. Robins's book has always been much admired, which relates to the experimental method of ascertaining the actual velocities of shot, and in imitation of which, but on a large scale, those experiments were made which were described in my paper. Experiments in the manner of Mr. Robins were generally repeated by his commentators, and others, with universal satisfaction; the method being so just in theory, so simple in practice, and altogether so ingenious, that it immediately gave the fullest conviction of its excellence, and the eminent abilities of the inventor. The use which our author made of his invention, was to obtain the real velocities of bullets experimentally, that he might compare them with those which he had computed *a priori* from a new theory of gunnery which he had invented, in order to verify the principles on which it was founded. The success was fully answerable to his expectations, and left no doubt of the truth of his theory, at least when applied to such pieces and bullets as he had used. These however were but small, being only musket balls of about an ounce weight: for, on account of the great size of the machinery necessary for such experiments, Mr. Robins, and other ingenious gentlemen, have not ventured to extend their practice beyond bullets of that kind, but contented themselves with ardently wishing for experiments to be made in a similar manner with balls of a larger sort. By the experiments described in my paper therefore I endeavoured, in some degree, to supply that defect, having used cannon balls of above twenty times the size, or from one pound to near three pounds weight. Those are the only experiments, that I know of, which have been made in that way with cannon balls, although the conclusions to be deduced from such a course, are of the greatest importance in those parts of natural philosophy which are connected with the effects of fired gunpowder: nor do I know of any other practical method besides that above, of ascertaining the initial velocities of military projectiles within any tolerable degree of the truth; except that of the recoil of the gun, hung on an axis in the same manner as the pendulum; which was also first pointed out and used by Mr. Robins, and which has lately been practised also by Benjamin Thompson,

Thompson, Esq. in his very ingenious and accurate set of experiments with musket balls, described in his paper in the Philosophical Transactions for the year 1781. The knowledge of this velocity is of the greatest consequence in gunnery: by means of it, together with the law of the resistance of the medium, every thing is determinable which relates to that business; for, as I remarked in the paper above-mentioned on my first experiments, it gives us the law relative to the different quantities of powder, to the different weights of balls, and to the different lengths and sizes of guns, and it is also an excellent method of trying the strength of different sorts of powder. Beside these, there does not seem to be any thing wanting to answer every inquiry that can be made concerning the flight and ranges of shot, except the effects arising from the resistance of the medium.

2. In that course of experiments were compared the effects of different quantities of powder, from two to eight ounces; the effects of different weights of shot; and the effects of different sizes of shot, or different degrees of windage, which is the difference between the diameter of the shot and the diameter of the bore; all of which were found to observe certain regular and constant laws, as far as the experiments were carried. And at the end of each day's experiments, the deductions and conclusions were made, and the reasons clearly pointed out why some cases of velocity differ from others, as they properly and regularly ought to do. So that I am surprized how they could be misunderstood by Mr. Templehof, captain in the Prussian artillery, when speaking of the irregularities in such experiments, he says, (page 126 of *Le Bombardier Prussien*, printed at Berlin, 1781) "*La meme chose arriva a Mr. Hutton, il la trouva de 626 pieds, & le jour suivant de 973 pieds, tout les circonstances étant d'ailleurs égales:*" which last words shew that Mr. T. had either misunderstood, or had not read the reason, which is a very sufficient one, for this remarkable difference: it is expressly remarked in page 71 of my paper in the Philosophical Transactions,

Transactions, that *all the circumstances* were *not* the same, but that the one ball was much smaller than the other, and that it had the less degree of velocity, 626 feet, because of the greater loss of the elastic fluid by the windage in the case of the smaller ball. On the contrary, the velocities in those experiments were even more uniform and similar than could be expected in such large machinery, and in a first attempt of the kind too. And from the whole, the following important conclusions were fairly drawn and stated, viz.

“(1.) And first, it is made evident by these experiments, that powder fires almost instantaneously, seeing that almost the whole of the charge fires, though the time be much diminished.

“(2.) The velocities communicated to shot of the same weight, with different quantities of powder, are nearly in the subduplicate ratio of those quantities. A very small variation, in defect, taking place when the quantities of powder become great.

“(3.) And when shot of different weights are fired with the same quantity of powder, the velocities communicated to them, are nearly in the reciprocal sub-duplicate ratio of their weights.

“(4.) So that, universally, shot which are of different weights, and impelled by the firing of different quantities of powder, acquire velocities which are directly as the square roots of the quantities of powder, and inversely as the square roots of the weights of the shot, nearly.

“(5.) It would therefore be a great improvement in artillery, to make use of shot of a long form, or of heavier matter; for thus the momentum of a shot, when fired with the same weight of powder, would be increased in the ratio of the square root of the weight of the shot.

“(6.) It

“ (6.) It would also be an improvement, to diminish the windage: for, by so doing, one third or more of the quantity of powder might be saved.

“ (7.) When the improvements mentioned in the last two articles are considered as both taking place, it is evident that about half the quantity of powder might be saved; which is a very considerable object. But important as this saving may be, it seems to be still exceeded by that of the guns: for thus a small gun may be made to have the effect and execution of one of two or three times its size in the present way, by discharging a long shot of two or three times the weight of its natural ball, or round shot: and thus a small ship might discharge shot as heavy as those of the greatest now made use of.

“ Finally, as the above experiments exhibit the regulations with regard to the weight of powder and balls, when fired from the same piece of ordnance; so by making similar experiments with a gun, varied in its length, by cutting off from it a certain part before each course of experiments, the effects and general rules for the different lengths of guns, may be certainly determined by them. In short, the principles on which these experiments were made, are so fruitful in consequences, that, in conjunction with the effects of the resistance of the medium, they seem to be sufficient for answering all the inquiries of the speculative philosopher, as well as those of the practical artillerist.”

3. Such then was the state of the first set of experiments with cannon balls in the year 1775, and such were the probable advantages to be derived from them. I do not however know that any use has hitherto been made of them by authority for the public service; unless perhaps we are to except the instance of Carronades, a species of ordnance which hath since been invented, and in some degree adopted in the public service



service ; for in this instance the proprietors of those pieces, by availing themselves of the circumstances of large balls, and very small windage, with small charges of powder, have been able to produce very considerable and useful effects with those light pieces, at a very small expence. Or perhaps those experiments were too much limited, and of too private a nature, to merit a more general notice. Be that however as it may, the present additional course, which is to make the subject of this tract, will have very great advantages over the former, both in point of extent, variety, improvements in machinery, and in authority. His Grace the Duke of Richmond, the present master-general of the ordnance, in his indefatigable endeavours for the good of the public service, was pleased to order this extensive course of experiments, and to give directions for providing guns, and machinery, and every thing compleat and fitting for the proper execution of them.

4. This course of experiments has been carried on under the direction of Major Blomefield, inspector of artillery, an officer of great professional merit, and whose ingenious contrivances in the machinery do him great credit. It has been our employment for three successive summers, namely, those of the years 1783, 1784, and 1785 ; and indeed it might be continued still much longer, either by extending it to more objects, or to more repetitions of experiments for the same object.

5. The objects of this course have been various. But the principal articles of it as follows :

(1.) The velocities with which balls are projected by equal charges of powder, from pieces of the same weight and calibre, but of different lengths.

(2.) The velocities with different charges of powder, the weight and length of the gun being the same.

3. The

(3.) The greatest velocity due to the different lengths of guns, to be obtained by increasing the charge as far as the resistance of the piece is capable of sustaining.

(4.) The effect of varying the weight of the piece; every thing else being the same.

(5.) The penetration of balls into blocks of wood.

(6.) The ranges and times of flight of balls; to compare them with their initial velocities, for determining the resistance of the medium.

(7.) The effect of wads;
of different degrees of ramming, or compressing the charge;
of different degrees of windage;
of different positions of the vent;
of chambers, and trunnions, and every other circumstance necessary to be known for the improvement of artillery.

Of the Nature of the Experiment, and of the Machinery used in it.

6. THE effects of most of the circumstances last mentioned are determined by the actual velocity with which the ball is projected from the mouth of the piece. Therefore the primary object of the experiments is, to discover that velocity in all cases, and especially in such as usually occur in the common practice of artillery. This velocity is very great; from one thousand to two thousand feet or more, in a second of time. For conveniently estimating so great a velocity, the first thing necessary is, to reduce it, in some known proportion, to a small one.

Which we may conceive to be effected in this manner : suppose the ball, projected with a great velocity, to strike some very heavy body, such as a large block of wood, from which it will not rebound, so that after the stroke they may both proceed forward together with a common velocity. By this means, it is obvious that the original velocity of the ball may be reduced in any proportion, or to any slow velocity which may conveniently be measured, by making the body struck to be sufficiently large : for it is well known that the common velocity, with which the ball and the block of wood would move on together after the stroke, bears to the original velocity of the ball before the stroke, the same ratio which the weight of the ball has to that of the ball and block together. Thus then velocities of one thousand feet in a second are easily reduced to those of two or three feet only : which small velocity being measured by any convenient means, let the number denoting it be increased in the ratio of the weight of the ball to the weight of the ball and block together, and the original velocity of the ball itself will thereby be obtained.

7. Now this reduced velocity is rendered easy to be measured by a very simple and curious contrivance, of Mr. Robins, which is this : the block of wood, which is struck by the ball, instead of being left at liberty to move straight forward in the direction of the motion of the ball, is suspended, like the weight of the vibrating pendulum of a clock, by a strong iron stem, having a horizontal axis at the top, on the ends of which it vibrates freely when struck by the ball. The consequence of this simple contrivance is evident : this large ballistic pendulum, after being struck by the ball, will be penetrated by it to a small depth, and it will then swing round its axis, describing an arch, which will be greater or less according to the force of the blow struck ; and from the magnitude of the arch described by the vibrating pendulum, the velocity of any point of the pendulum can be easily computed : for a body acquires the same velocity by falling from the same height,

whether it descend perpendicularly down, or otherwise; therefore, having given the length of the arc described by the center of oscillation, and its radius, the versed sine becomes known, which is the height perpendicularly descended by that point of the pendulum. The height descended being thus known, the velocity acquired in falling through that height becomes known also, from the common rules for the descent of bodies by the force of gravity. And the velocity of this center, thus obtained, is to be esteemed the velocity of the whole pendulum itself; which being now given, that of the ball before the stroke becomes known, from the given weights of the ball and pendulum. Thus then the determination of the very great velocity of the ball is reduced to the mensuration of the magnitude of the arch described by the pendulum, in consequence of the blow struck.

8. Now this arch may be determined in various ways: in the following experiments it was ascertained by measuring the length of its chord, which is the most useful line about it for making the calculation by; and this chord was measured sometimes by means of a piece of tape or narrow ribbon, the one end of which was fastened to the bottom of the pendulum, and the rest of it made to slide through a small machine contrived for the purpose; and sometimes it was measured by the trace of the fine point of a stylette in the bottom of the pendulum, made in an arch concentric with the axis, and covered with a composition of a proper consistence; which will be particularly described hereafter.

9. Another similar method of measuring the great velocity of the ball is, by observing the arch of recoil of the gun, when it is hung also after the manner of a pendulum: for, by loading the gun with adventitious weight, it may be made so heavy as to swing any convenient extent of arch we please, which arch it is evident will be greater or less according to the velocity of the ball, or force of the inflamed powder, since action and re-action are equal and contrary; that is, the velocity of the ball will

be greater than the velocity of the center of oscillation of the gun, in the same proportion as the weight of the gun exceeds the weight of the ball. And therefore, if the velocity of the center of oscillation of the gun be computed, from the chord of the arc described by it in the recoil, the velocity of the ball will be found by this proportion; namely, as the weight of the ball is to the weight of the gun, so is the velocity of the gun to the velocity of the ball: that is, if the weight of powder had no effect on the recoil.

10. This description may suffice to convey a general idea of the nature and principles of the experiment, for determining the velocity with which a ball is projected, by any charge of powder, from a piece of ordnance. But it is to be observed that, besides the center of oscillation, and the weights of the ball and pendulum, or gun, the effect of the blow depends also on the place of the center of gravity in the pendulum or gun, and that of the point struck, or the place where the force is exerted; for it is evident that the arch of vibration will be greater or less according to the situation of these two points also. It will therefore be necessary now to give a more particular description of the machinery, and of the methods of finding the aforesaid requisites; and then we shall investigate our general rules for determining the velocity of the ball, in all cases, from them and the chord of the arch of vibration, either of the pendulum or gun.

Of the Guns, Powder, Balls, and Machinery employed in these Experiments.

11. FIVE very fine brass one-pounder guns were cast and prepared, in Woolwich Warren, for these experiments, and bored as true as possible; the common diameter of their bore being 2 inches and $\frac{2}{100}$ parts.

parts of an inch. These five guns are exactly represented in plate 1, with the scale of their dimensions, by which they were drawn. Three of these, namely, n°. 1, 2, 3, are nearly of the same weight, but of the respective lengths of 15, 20, and 30 calibers; in order to ascertain the effect of different lengths of bore, with the same weight of gun, powder, and ball. The other two, n°. 4 and 5, were heavier, and of 40 calibers in length; to obtain the effects of the longest pieces. N°. 5 was more expressly to shew the effect of different lengths of the same gun: and for this purpose, it was to be fired a sufficient number of rounds with its whole length; and then to be successively diminished, by sawing off it 6 or 12 inches at a time, till it should be all cut away: firing a number of rounds with it at each length. And for the convenience of suspending this gun near its center of gravity for all the different lengths of it, a long thin slip was cast with it, extending along the under side of it, from the breech to almost the middle of its length. By perforating this slip through with holes immediately under the center of gravity for each length, after being cut, a bolt was to pass through the hole, on which the gun might be suspended. The other guns were slung by their trunnions.

The exact weight and dimensions of all these guns are exhibited in the following table.

N°. of the gun	Length of the				Diameter at the		Diam. of the bore	Weight
	Piece, in		Bore, in		breech	muzzle		
	calib.	inch	calib.	inch	inch	inch	inch	lb.
1	15	30.3	13.91	28.2	7.85	6.88	2.02	290
2	19.98	40.35	18.86	38.1	7.43	5.92	2.02	289
3	29.2	60	28.4	57.37	6.73	4.68	2.02	295
4	41.04	82.9	39.55	79.9	6.1	4.31	2.02	378
5	40.84	82.5	39.83	80.47	6.47	4	2.02	502

12. As these guns were to be flung by their trunnions, to observe the relation between the velocity of the ball and the arch of recoil described by the gun, vibrating on an axis, certain leaden weights were cast, to fit on very exactly about the trunnions of the gun, to render it so heavy, as that the arch of recoil might not be inconveniently great. These consisted, first of central pieces to fit the trunnions, and then over them cylindrical rings of different sizes, both turned to fit exactly; the whole being held firmly together by iron bolts put into holes bored through all the pieces. These were also of different sizes, so as to bring all the guns exactly up to the same weight; the whole weight of each, together with 188 lb weight of iron, about the stem and machine, by which the gun was flung, was 917 lb; with which weight most of the experiments were made: notice being always taken when any alteration was made in the weights, as well as in the other circumstances. The common weight of 917 lb is made up of the different guns and leads, and the common weight of iron, as below:

N ^o .	Guns	Leads	Iron	Total
1 - - -	290	+ 439	+ 188	= 917
2 - - -	289	+ 440	+ 188	= 917
3 - - -	295	+ 434	+ 188	= 917
4 - - -	378	+ 351	+ 188	= 917
5 - - -	502	+ 227	+ 188	= 917

These were the weights at first; but soon after, the braces, or strengthening rods of the gun frame, were made longer and thicker, which added 11 lb to their weights, and then the whole weight of each was 928 lb.

13. In these experiments, the velocity of the ball, by which the force of the powder is determined, was to be measured both by the ballistic pendulum into which the ball was fired, and by the arch of recoil of the gun, which was hung on an axis by an iron stem, after the same manner

manner as the pendulum itself, and the arcs vibrated in both cases measured in the same way. Plates II and III contain general representations of the machinery of both; namely, a side view and a front view of each, as they hung by their stem and axis on the wooden supports. In plate II, fig. 1 is the side-view of the pendulum, and fig. 2 the side-view of the gun, as flung in their frames. And in plate III, fig. 1 and 2 are the front-views of the same.

14. In fig. 1, of both plates, A is the pendulous block of wood, into which the balls are fired, strongly bound with thick bars of iron, and hung by a strong iron stem, which is connected by an axis at top; the whole being firmly braced together by crossing diagonal rods of iron. The cylindrical ends of the axis, both in the gun and pendulum, were at first placed to turn upon smooth flat plate-iron surfaces, having perpendicular pins put in before and behind the sides of the axis, to keep it in its place, and prevent it from slipping backwards and forwards. But, this method being attended with too much friction, the ends of the axis were supported and made to roll upon curved pieces, having the convexity upwards, and the pins, before and behind the axis, set so as not quite to touch it; which left a small degree of play to the axis, and made the friction less than before. But, still farther to diminish the friction, the lower side of the ends of the axis was sharpened off a little, something like the axis of a scale beam, and made to turn in hollow grooves, which were rounded down at both ends, and standing higher in the middle, like the curvature of a bent cylinder; by which means the edge of the axis touched the grooves, not in a line, but in one point only; when it vibrated with very great freedom, having an almost imperceptible degree of friction. The several times and occasions when these, and other improvements, were introduced and used, will be more particularly noticed in the journal of the experiments.

15. At

15. At first, the chord of the arc, of vibration and recoil, was measured by means of a prepared narrow tape, divided into inches and tenths, as before. A new contrivance of machinery was however made for it. From the bottom of the pendulum, or gun-frame, proceeded a tongue of iron, which was raised or lowered by means of a screw at *B*; this was cloven at the bottom *C*, to receive the end of the tape, and the lips then pinched together by a screw, which held the tape fast. Immediately below this the tape was passed between two slips of iron, which could be brought to any degree of nearness by two screws; these pieces were made to slide vertically up and down a groove in a heavy block of wood, and fixed at any height by a screw *D*. One of these latter pieces was extended out a considerable length, to prevent the tape from getting over its ends, and entangling in the returns of the vibrations. The extent of tape drawn out in a vibration, it is evident, is the chord of the arc described, and counted in inches and tenths, to the radius measured from the middle of the axis to the bottom of the tongue.

16. This method however was found to be attended with much trouble, and many inconveniences, as well as doubts and uncertainty sometimes. For which reasons we afterwards changed this method of measuring the chord of vibration for another, which answered much better in every respect. This consisted in a block of wood, having its upper surface *E F* formed into a circular arc, whose center was in the middle of the axis, and consequently its radius equal to the length from the axis to the upper surface of the block. In the middle of this arch was made a shallow groove of 3 or 4 inches broad, running along the middle, through the whole length of the arch. This groove was filled with a composition of soft-soap and wax, of about the consistence of honey, or a little firmer, and its upper side smoothed off even with the general surface of the broad arch. A sharp spear or stylette then proceeded from the bottom of the pendulum or gun-frame, and so low as just to enter and scratch along the surface of the composition in the groove, without
having

having any sensible effect in retarding the motion of the body. The trace remaining, the extent of it could easily be measured. This measurement was effected in the following manner:—A line of chords was laid down upon the upper surface of the wooden arch, on each side of the groove, and the divisions marked with lines on a ground of white paint: the edge of a straight ruler being then laid across by the corresponding divisions, just to touch the farthest extent of the trace in the composition, gave the length of the chord as marked on the arch. To make the computations by the rule for the velocity easier, the divisions on the chords were made exact thousandth parts of the radius, which saved the trouble of dividing by the radius at every operation. The manner in which I constructed this line of chords on the face of the arch was this: The radius was made just 10 feet; I therefore prepared a smooth and straight deal rod, upon which I set off 10 feet; I then divided each foot into 10 equal parts, and each of these into 10 parts again; by which means the whole rod or radius was divided into 1000 equal parts, being 100th parts of a foot. I then transferred the divisions of the rod to the face of the arch in this manner, namely; the first division of the rod was applied to the side of the arch at the beginning of it, and made to turn round there as a center; then, in that position, the rod, when turned vertically round that point, always touched the side of the arch, and the divisions of it were marked on the edge of the arch, successively as they came into a coincidence with it.

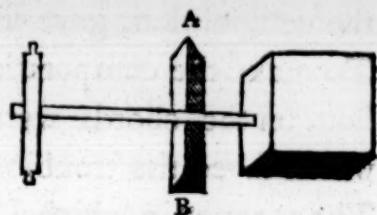
17. In fig. 2, plate 11, G shews the leaden weights placed about the trunnions; H a screw for raising or depressing the breech of the gun, by means of the piece I embracing the cascable, and moveable along the perpendicular arm KL, to suit the different lengths of guns, and held to it by a screw passing through the slit made along it.

The machines and operations for finding the ranges will be described hereafter.

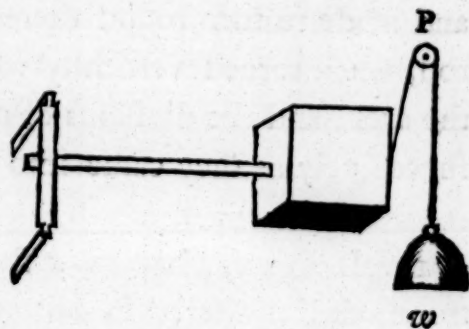
Of the Centers of Gravity and Oscillation.

18. It being necessary to know the position of the centers of gravity and oscillation, without which the velocity cannot be computed; these were commonly determined every day as follows:

The center of gravity was found by one or both of these two methods. First, a triangular prism of iron *AB*, being placed on the ground with one edge upwards, the pendulum or gun-frame was laid across it, and moved backward or forward, on the stem or block, as the case required, till the two parts exactly balanced each other in a horizontal position. Then, as it lay, the distance was measured from the middle of the axis to the part which rested on the edge of the prism, or the place of the center of gravity, which is the distance *g* of that center below the axis.



19. The other method is this: The ends of the axis being supported on fixed uprights, and a chord fastened to the lower end of the block, or of the gun frame, and passed over a pulley at *P*, different weights *w* were fastened to the other end of it, till the body was brought to a horizontal position. Then, taking also the whole weight of the body, and its length from the axis to the bottom, where the chord was fixed, the place of the center of gravity is found by this proportion:



As *p* the weight of the pendulum :
 is to *w* the appended weight ::
 so is *d* the whole length from the axis to the chord :
 to $\frac{dw}{p}$ the distance from the axis to the center of gravity.

Either

Either of these two methods gave the place of the center of gravity sufficiently exact; but the agreement of the results of both of them was still more satisfactory.

20. To find the center of oscillation, the ballistic pendulum, or the gun, was hung up by its axis in its place, and then made to vibrate in small arcs, for 1 minute, or 2, or 5, or 10 minutes; the more the better; as determined either by a half second pendulum, or a stop watch, or a peculiar time-piece, measuring the time to 40th parts of a second; and the number of vibrations performed in that time carefully counted. Having thus obtained the time answering to a certain number of vibrations, the center of oscillation is easily found: for if n denote the number of vibrations made in s seconds, and l the length of the second pendulum, then it is well known that $n^2 : s^2 :: l : \frac{s^2 l}{n^2}$ the distance from the axis of motion to the center of oscillation. And here if s be 60 seconds, or one minute, and n the number of vibrations performed in 1 minute, as found by dividing the whole number of vibrations, actually performed, by the whole number of minutes; then is $n^2 : 60^2 :: l : \frac{3600l}{nn}$ the distance to the center of oscillation. But, by the best observations on the vibration of pendulums, it is found that $l = 39\frac{1}{8}$ inches is the length of the second pendulum for the latitude of London, or of Woolwich; and therefore $\frac{3600 \times 39\frac{1}{8}}{nn}$ or $\frac{140850}{nn} = o$, will be the distance, in inches, or $= \frac{11737.5}{nn}$ in feet, of the center of oscillation below the axis. And by this rule the place of that center was found for each day of the experiments.

Of the Rule for Computing the Velocity of the Ball.

21. Having described the methods of obtaining the necessary dimensions and weights, I proceed now to the investigation of the theorem by which

which the velocity of the ball is to be computed: and first by means of the pendulum.

The several weights and measures being found, let

b denote the weight of the ball,

p the weight of the pendulum,

g the distance to its center of gravity,

o the distance to its center of oscillation,

i the distance to the point of impact, or point struck,

c the chord of the arch described by the pendulum,

r its radius, or distance to the tape or arch,

v the initial or original velocity of the ball.

Then, from the nature of oscillatory motion, bii will express the sum of the forces of the ball acting at the distance i from the axis, and pgo the sum of the forces of the pendulum, and consequently $pgo + bii$ the sum for both the ball and pendulum together; and if each be multiplied by its velocity, $bii v$ will be the quantity of motion of the ball, and $(pgo + bii) \times z$ the quantity for the pendulum and ball together; where z is the velocity of the point of impact. But these quantities of motion, before and after the blow, must be equal to each other, therefore $(pgo + bii) \times z = bii v$, and consequently $z = \frac{bii v}{pgo + bii}$ is the velocity of the point of impact. Now because of the accession of the ball to the pendulum, the place of the center of oscillation will be changed; and the distance y of the new or compound center of oscillation will be found by dividing $pgo + bii$ the sum of the forces, by $pg + bi$ the sum of the momenta, that is $y = \frac{pgo + bii}{pg + bi}$ is the distance of the new or compound center of oscillation below the axis. Then, because $\frac{bii v}{pgo + bii}$ is the velocity of the point whose distance is i , by similar figures we shall have this proportion, as $i : \frac{pgo + bii}{pg + bi}$ (or y) :: $\frac{bii v}{pgo + bii} : \frac{bi v}{pg + bi}$ the velocity of this compound center of oscillation.

Again,

Again, by the property of the circle, $2r : c :: c : \frac{cc}{2r}$, which will be the versed sine of the described arc, to the chord c and radius r ; and hence, by similar figures, $r : y$ or $\frac{pgo + bii}{pg + bi} :: \frac{cc}{2r} : \frac{cc}{2rr} \times \frac{pgo + bii}{pg + bi}$ the corresponding versed sine to the radius y , or the versed sine of the arc described by the compound center of oscillation; which call v . Then, because the velocity lost in ascending through the circular arc, or gained in descending through the same, is equal to the velocity acquired in descending freely by gravity through its versed sine, or perpendicular height, therefore the velocity of this center of oscillation will also be equal to the velocity generated by gravity in descending through the space v or $\frac{cc}{2rr} \times \frac{pgo + bii}{pg + bi}$. But the space described by gravity in one second of time, in the latitude of London, is 16.09 feet, and the velocity generated in that time 32.18; therefore, by the nature of free descents, $\sqrt{16.09} : \sqrt{v} :: 32.18 : \frac{5.6727c}{r} \sqrt{\frac{pgo + bii}{pg + bi}}$, the velocity of the same center of oscillation, as deduced from the chord of the arc which is actually described.

Having thus obtained two different expressions for the velocity of this center, independent of each other, let an equation be made of them, and it will express the relation of the several quantities in the question: thus then we have $\frac{biw}{pg + bi} = \frac{5.6727c}{r} \sqrt{\frac{pgo + bii}{pg + bi}}$. And from this equation we get $v = \frac{5.6727c}{bir} \sqrt{(pgo + bii) \times (pg + bi)}$ the true expression for the original velocity of the ball the moment before it strikes the pendulum. And this theorem agrees with those of Messrs. Euler and Antoni, and also with that of Mr. Robins nearly, for the same purpose, when his rule is corrected by the paragraph which was by mistake omitted in his book when first published; which correction he himself gave in a paper in the Philosophical Transactions for April 1743, and where he informs us that all the velocities of balls, mentioned in his book, except the first only, were

were computed by the corrected rule. Though the editor of his works, published in 1761, has inadvertently neglected this correction, and printed his book without taking any notice of it. And that remark, had M. Euler observed it, might have saved him the trouble of many of his animadversions on Mr. Robins's work.

22. But this theorem may be reduced to a form much more simple and fit for use, and yet be sufficiently near the truth. Thus, let the root of the compound factor $(pgo + bii) \times (pg + bi)$ be extracted, and it will be equal to $(pg + bi \cdot \frac{o+i}{2o}) \times \sqrt{o}$, within the 100000th part of the true value, in such cases as commonly happen in practice. But since $bi \cdot \frac{o+i}{2o}$, in our experiments, is usually but about the 500th, or 600th, or 800th part of pg , and since bi differs from $bi \cdot \frac{o+i}{2o}$ only by about the 100th part of itself, therefore $pg + bi$ is within the 50000th part of $pg + bi \cdot \frac{o+i}{2o}$. Consequently $v = 5.6727c \cdot \frac{pg + bi}{bir} \sqrt{o}$ very nearly. Or, farther, if g be written for i in the last term bi , then finally $v = 5.6727gc \cdot \frac{p+b}{bir} \sqrt{o}$; which is an easy theorem to be used on all occasions; and being within the 5000th part of the true quantity, it will always give the velocity true within less than half a foot, even in the cases of the greatest velocity. Where it must be observed, that c, g, i, r , may be taken in any measures, either feet or inches, &c. provided they be but all of the same kind; but o must be in feet, because the theorem is adapted to feet.

23. As the balls remain in the pendulum during the time of making one whole set of experiments, both its weight and the position of the centers of gravity and oscillation will be changed by the addition of each ball which is lodged in the wood; and therefore p, g, o must be corrected after every shot, in the theorem for determining the velocity v . Now the

the succeeding value of p is always $p + b$; or p is to be corrected by the continual addition of b : and the succeeding value of g is $g + \frac{i-g}{p+b}b$, or $g + \frac{i-g}{p}b$ nearly; or g is corrected by adding always $\frac{i-g}{p}b$ to the next preceding value of g : and lastly, o is to be corrected by taking for its new values successively $\frac{pgo + bii}{pg + bi}$, or by adding always $\frac{i-o}{pg + bi}bi$, or $\frac{i-o}{p}b$ nearly, to the preceding value of o : so that the three corrections are made by adding always,

b to the value of p ,

$\frac{i-g}{p} \times b$ to the value of g ,

$\frac{i-o}{p} \times b$ to the value of o .

That is, when b is very small in respect of p .

24. But as the distance of the center of oscillation o , whose square root is concerned in the theorem for the velocity v , is found from the number of vibrations n performed by the pendulum; it will be better to substitute, in that theorem, the value of o in terms of n . Now by Art. 20, the value of o is $\frac{11737.5}{nn}$ feet, and consequently $\sqrt{o} = \frac{108.3398}{n}$; which value of $\sqrt{o}$ being substituted for it in the theorem $v = 5.6727gc \times \frac{p+b}{bir} \sqrt{o}$, it becomes $v = 614.58gc \times \frac{p+b}{birn}$, or $= \frac{59000}{96} \times \frac{p+b}{birn} gc$, the simplest and easiest formula for the velocity of the ball in feet: where c, g, i, r may be taken in any one and the same measure, either all inches, or all feet, or any other measure.

25. It will be necessary here to add a correction for n instead of that for o in Art. 23. Now, the correction for o being $\frac{i-o}{pg + bi}bi$, and the value

value of $n = \frac{375.3}{\sqrt{o}}$ inches, the correction for n will be

$$375.3 \times \left(\frac{1}{\sqrt{o}} - \frac{1}{\sqrt{o + \frac{i-o}{pg+bi} \cdot bi}} \right) = 375.3 \times \left(\frac{1}{\sqrt{o}} - \frac{1}{\sqrt{o} \times \sqrt{\frac{i-o}{pg+bi} \cdot \frac{bi}{o}}} \right)$$

$$= n - \frac{n}{\sqrt{1 + \frac{i-o}{pg+bi} \cdot \frac{bi}{o}}} = n - \frac{n}{1 + \frac{i-o}{pg+bi} \cdot \frac{bi}{2o}} \text{ nearly} =$$

$\frac{bin \times (inn - 140850)}{281700pg + bi \times (inn + 140850)}$ by substituting the value of o instead of it:

Which correction is negative, or to be subtracted from the former value of n . The corrections for p and g being b and $\frac{i-g}{p+b}b$, as in Art. 23; which are both additive. But the signs of these quantities must be changed when b is negative.

26. Before we quit this rule, it may be necessary here to advert to three or four circumstances which may seem to cause some small error in the initial velocity, as determined by the formula in Art. 24. These are the friction on the axis, the resistance of the air to the back of the pendulum, the time which the ball employs in penetrating the wood of the pendulum, and the resistance of the air to the ball in its passage between the gun and the pendulum.

As to the first of these, namely, the friction on the axis, by which the extent of its vibration is somewhat diminished; it may be observed, that the effect of this cause can never amount to a quantity considerable enough to be brought into account in our experiments; for, besides that care was taken to render this friction as small as possible, the effect of the small part which does remain is nearly balanced by the effect it has on the number n of vibrations performed in a minute; for the friction on the axis will a little retard its motion, and cause its vibrations to be slower, and fewer; so that c the length of a vibration, and n the number of vibrations, being both diminished by this cause, nearly in an equal degree, and c being a multiplier, and n a divisor, in our formula, it is evident

evident that the effect of the friction in the one case operates against that in the other, and that the difference of the two is the real disturbing cause, and which therefore is either equal to nothing, or very nearly so.

27. The second cause of error is the resistance of the air against the back of the pendulum, by which its motion is somewhat impeded. This resistance hinders the pendulum from vibrating so far, and describing so large an arch, as it would do if there was no such resistance; therefore the chord of the arc which is actually described and measured, is less than it really ought to be; and consequently the velocity of the ball, which is proportional to that chord, will be less than the real velocity of the ball at the moment it strikes the pendulum. And although the pendulum be very heavy, and its motion but slow, and consequently the resistance of the air against it very small, it will yet be proper to investigate the real effect of it, that we may be sure whether it may safely be neglected or not.

In order to this, let the annexed figure represent the back of the pendulum, moving on its axis; and put

p = weight of the pendulum,

a = DE its breadth,

r = AB the distance to the bottom,

e = AC the distance to the top,

x = AF any variable distance,

g = distance of the center of gravity,

o = distance of the center of oscillation,

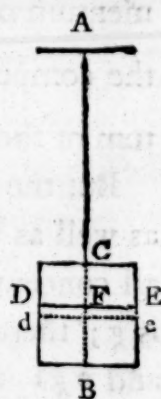
v = velocity of the center of oscillation, in any part of the vibration,

b = 16.09 feet, the descent of gravity in 1 second,

c = the chord of the arc actually described by the center of oscillation,

and

c = the chord which would be described by it if the air had no resistance.



Then $o : x :: v : \frac{vx}{o}$ the velocity of the point F of the pendulum; and $4b^2 : b :: \frac{v^2 x^2}{o^2} : \frac{v^2 x^2}{4bo^2}$ the height descended by gravity to generate the velocity $\frac{vx}{o}$. Now the resistance of the air to the line DFE is equal to the pressure of a column of air upon it, whose height is the same $\frac{v^2 x^2}{4bo^2}$, and therefore that pressure or weight is $\frac{nav^2 x^2}{4bo^2}$, where n is the specific gravity, or weight of one cubic measure of air, or $n = \frac{62\frac{1}{2}}{850} \text{ lb} = \frac{5}{68} \text{ lb}$. Hence then $\frac{nav^2 x^2}{4bo^2}$ is the pressure on DEd, and $\frac{nav^2 x^3}{4bo^2}$ the momentum of the pressure on the same DE, or the fluxion of the momentum on the block of the pendulum; and the correct fluent gives $na \times \frac{r^4 - e^4}{16bo^2} v^2$ for the momentum of the air on the whole pendulum, supposing that on the stem AC to be nothing, as it is nearly, both on account of its narrowness, and the diminution of the momentum of the particles by their nearness to the axis. Put now $A =$ the compound coefficient $na \times \frac{r^4 - e^4}{16bo^2}$, so shall Av^2 denote the momentum of the air on the back of the pendulum.

But the motion of the pendulum is also obstructed by its own weight, as well as by the resistance of the air; and that weight acts as if it were all centered in the center of gravity, whose distance below the axis is g ; therefore pg is its momentum in its natural or vertical direction, and pgs its momentum perpendicular to the motion of the pendulum, when s is the sine of the angle which it makes at any time with the vertical position, to the radius 1. Hence $pgs + Av^2$ is the momentum of both the resistances together, namely that of the pressure of the air, and of the weight of the pendulum. And consequently $\frac{pgs + Av^2}{pg} = s + \frac{A}{pg} v^2$ is the real retarding force to the motion of the pendulum, at the center of oscillation; which force call f .

Now if z denote the arc described by the center of oscillation, when its velocity is v , or $\frac{z}{o}$ the arc whose sine is s ; we shall have

$$\dot{z} = \frac{\dot{o}s}{\sqrt{1-s^2}}, \text{ and, by the doctrine of forces,}$$

$$v\dot{v} = -2bf\dot{z} = -2bo \cdot \frac{s\dot{s}}{\sqrt{1-s^2}} - \frac{2boA}{pg} \cdot \frac{v^2\dot{s}}{\sqrt{1-s^2}}.$$

But $\frac{cc}{2o}$ is the versed sine or height of the whole arc whose chord is c , and $o \times (1 - \sqrt{1-s^2}) = o - o\sqrt{1-s^2}$ is the versed sine or height of the part whose sine is os , therefore $\frac{cc}{2o} - o + o\sqrt{1-s^2}$ is their difference, or the height of the remaining part, and is nearly equal to the height due to the velocity v ; therefore

$$b : 4bb :: \frac{cc}{2o} - o + o\sqrt{1-s^2} : v^2 = 4b \times (\frac{cc}{2o} - o + o\sqrt{1-s^2})$$

nearly. Then by substituting this for v^2 in the value of $v\dot{v}$, we have

$$v\dot{v} = -2bo \cdot \frac{s\dot{s}}{\sqrt{1-s^2}} - \frac{8bb o A}{pg} \times (\frac{c^2 - 2o^2}{2o} \cdot \frac{s\dot{s}}{\sqrt{1-s^2}} + o\dot{s});$$

and the fluents give

$$v^2 = 4bo\sqrt{1-s^2} + \frac{16bb o A}{pg} (\frac{2o^2 - c^2}{2o^2} z - os) + Q;$$

where Q is a constant quantity by which the fluent is to be corrected. Now, substituting v^2 for v^2 , and o for s , their corresponding values at the commencement of motion, the above fluent becomes

$$v^2 = 4bo + Q; \text{ from which the former subtracted, gives}$$

$$v^2 - v^2 = 4bo - 4bo\sqrt{1-s^2} - \frac{16bb o A}{pg} (\frac{2o^2 - c^2}{2o^2} z - os).$$

And when $v = o$, or the pendulum is at the full extent of its ascent, then

$$v^2 = 4bo - 4bo\sqrt{1-s^2} - \frac{16bb o A}{pg} (\frac{2o^2 - c^2}{2o^2} z - os),$$

at which point os is the sine of the whole arc whose chord is c , and consequently $s = \frac{c}{2o^2} \sqrt{4o^2 - c^2}$.

But the value of s being commonly small in respect of $\frac{c}{o}$, we shall have these following values nearly true, namely,

$$\sqrt{1 - ss} = 1 - \frac{1}{2}s^2 - \frac{1}{8}s^4 = 1 - \frac{4c^2 o^2 - c^4}{8o^4} - \frac{c^4}{8o^4} = 1 - \frac{c^2}{2o^2},$$

$$4bo - 4bo\sqrt{1 - ss} = 4bo \cdot \frac{c^2}{2o^2},$$

$$z = os + \frac{1}{6}os^3, \text{ and}$$

$$\frac{2o^2 - c^2}{2o^2}z - os = -\frac{c^2 s}{2o} + \frac{2o^2 - c^2}{12o}s^3,$$

which values, by substitution, give

$$v^2 = \frac{2bc^2}{o} + \frac{16b^2 o A}{pq} \left(\frac{c^2 s}{2o} - \frac{2o^2 - c^2}{12o}s^3 \right).$$

But $\frac{c^2}{2o}$ is the versed sine or height to the chord c , and $v^2 = 4b \cdot \frac{c^2}{2o} = \frac{2bc^2}{o}$ the square of the velocity due to that height; therefore

$$\frac{2bc^2}{o} = \frac{2bc^2}{o} + \frac{16b^2 o A}{pq} \left(\frac{c^2 s}{2o} - \frac{2o^2 - c^2}{12o}s^3 \right), \text{ and}$$

$$c^3 = c^3 + \frac{8boA}{pq} \left(\frac{c^2 s}{2} - \frac{2o^2 - c^2}{12}s^3 \right), \text{ or}$$

$$c^3 = c^3 + \frac{8bA}{pq} \left(\frac{c^3}{3} + \frac{c^5}{12o^2} \right), \text{ and}$$

$$c = c + \frac{4c^2 b A}{3pq} \text{ nearly, or substituting for } A,$$

$$c = c + \frac{nac^2}{12pg} \cdot \frac{r^4 - e^4}{o^2} = c \left(1 + \frac{nac}{12pg} \cdot \frac{r^4 - e^4}{o^2} \right).$$

So that the chord of the arc which is actually described, is to that which would be described if the air had no resistance, as 1 is to $1 + \frac{nac}{12pg} \cdot \frac{r^4 - e^4}{o^2}$; and therefore $\frac{nac}{12pg} \cdot \frac{r^4 - e^4}{o^2}$ is the part of the chord, and consequently of the velocity, lost by means of the resistance of the air. And the proportion is the same for the chords described by the lowest point, or any other point, of the pendulum.

28. Now,

28. Now, to give an example, in numbers, of this effect of the resistance of the air; the ordinary mean values of the literal quantities are as here below,

namely,	therefore
$p = 700$	$nac = \frac{25}{102}$
$a = 2$	$12pg = 56000$
$r = 8\frac{1}{2}$	$r^4 = 5220$
$e = 6\frac{1}{2}$	$e^4 = 1785$
$g = 6\frac{2}{3}$	$r^4 - e^4 = 3435$
$o = 7\frac{1}{3}$	$o^2 = \frac{484}{9}$
$n = \frac{5}{68}$	$\frac{nac}{12pg} \cdot \frac{r^4 - e^4}{o^2} = \frac{1}{3577}$
$c = 1\frac{2}{3}$	



So that the part of the chord, or velocity, lost by this cause, namely, the resistance of the air on the back of the pendulum, is but about the $\frac{1}{3577}$, or about the $\frac{1}{4000}$ part of the whole; and therefore this effect scarcely ever amounts to so much as half a foot. Being indeed about $\frac{1}{2}$ of a foot when the velocity of the ball is 2000 feet,

$\frac{1}{4}$	-	when	-	it is	1000
$\frac{3}{4}$	-	when	-	it is	1500

and so on in proportion to the whole velocity of the ball.

And even this small effect may be supposed to be balanced by the method of determining the center of oscillation, or the number of vibrations made in a second. So that the number of oscillations, and the chord of the arc described, being both diminished by the resistance of the air; and the one of these quantities being a multiplier, and the other a divisor, in the formula for the velocity; the one of those small effects will nearly balance the other; much in the same way as the effects of the first cause, or the friction on the axis. So that, these effects may both of them be safely neglected, as in no case amounting to any sensible quantity.

In

In the beginning of this investigation, it is supposed that the resistance of the medium is equal to the weight of a column of the medium, whose base is the moving surface, and its altitude equal to that from whence a heavy body must fall to acquire the velocity of that surface. But some philosophers think the altitude should be only one half of that, and consequently the pressure only one half: which would render the resistance still less considerable. But if the altitude and resistance were even double of that above found, it might be still safely neglected.

28. The third seeming cause of error in our rule is the time in which the ball communicates its motion to the pendulum, or the time employed in the penetration. The principle on which the rule is founded supposes the momentum of the ball to be communicated in an instant; but this is not accurately the case, because this force is communicated during the time in which the ball makes the penetration. And although that time be evidently very small, scarcely amounting to the 500th part of a second, it will be proper to enquire what effect that circumstance may have on the truth of our theorem, or on the velocity of the ball, as computed by it.

In order to this, let the notation employed in Art. 21 be supposed here; and let ABC be a side-view of the pendulum moved out of the vertical position AD by the perpendicular blow of the ball against the point D or c . Also

let $x = DC$ the space moved by the point of impact c ,

$z = CB$ the depth penetrated by the ball,

$v =$ velocity of the ball at B ,

$u =$ velocity of the point c of the pendulum, and

$R =$ the uniform resisting force of the wood.

Then is $\frac{R}{b}$ the retarding force of the ball, which is constant. Again, as the motion of the pendulum arises from the resisting force R of the wood,



wood, Ri will be its momentum; and as the sum of the forces in the pendulum was found to be $= pgo$, the accelerating force of the point c will be $\frac{Rii}{pgo}$, which force is constant also. But in the action of forces that are constant, the time t is equal to the velocity divided by the force, and by $2b$ or 2×16.09 feet, and the space is equal to the square of the velocity divided by the force and by $4b$; consequently

$$t = \frac{pgou}{2biiR},$$

$$x = \frac{pgouu}{4biiR},$$

$$\text{and } t = \frac{-bv}{2bR},$$

$$x + z = \frac{-bv v}{4bR},$$

$$\text{or by correc. } t = \frac{b}{2bR} \times (v - v),$$

$$x + z = \frac{b}{4bR} \times (v^2 - v^2).$$

The two values of the time t being equated, we obtain

$$pgou = bii(v - v), \text{ or } pgou + bii v = bii v.$$

And when $v = u$, or the action of the ball on the pendulum ceases, this equation becomes $pgou + bii u = bii v$, and hence

$v = \frac{bii u}{pgou + bii}$ is the greatest velocity of the point c at the instant when the ball has penetrated to the greatest depth, and ceases to urge the pendulum farther. So that this velocity is the same, whatever the resisting force of the wood is, and therefore to whatever depth the ball penetrates, and the same as if the wood were perfectly hard, or the ball made no penetration at all. And this velocity of the point of impact also agrees with that which was found in Art. 21. So that the velocity communicated to the point of impact is the same, whether the impulse is made in an instant, or in some small portion of time. And hence, in the usual case of a penetration, because the block will have moved some small distance before it has attained its greatest velocity, it might at first view seem as if it would swing or rise higher than when that velocity is communicated in an instant, or when the pendulum is yet in its vertical position, and so might describe a longer chord, and shew a greater velocity of the ball than it ought. But on the other hand it must be considered,

considered, that in the small part of its swing, which the pendulum has made before the penetration is completed, or has attained its greatest velocity, just as much velocity will be lost by the opposing gravity or weight of the pendulum, as if it had set out from the vertical position with the said greatest velocity; and therefore the real velocity at that height will be the same in both cases. Hence then it may safely be concluded, that the circumstance of the ball's penetration causes no alteration in the velocity of it, as computed by our formula. And as it was before found that no sensible error is incurred by the two first circumstances, namely, the friction on the axis, and the resistance of the air to the back of the pendulum, we may be well assured that our formula brings out the true velocity with which the ball strikes the pendulum, without any sensible error.

29. Since $\frac{biiw}{pgo + bii}$ denotes the greatest velocity which the point c of the pendulum acquires by the stroke, dividing by i , we shall have $\frac{biw}{pgo + bii}$ for the angular velocity of the pendulum, or that of a radius 1. From which it appears that the vibration will be very small when i or the distance AD is small, and also when i is very great. And if we take this expression a *maximum*, and make its fluxion $= 0$, i only being variable, we shall obtain $pgo = bii$, and $i = \sqrt{\frac{pgo}{b}}$ for the distance of the center of percussion, or the point where the ball must strike so as to cause the greatest vibration in the pendulum; which point, in this case, is neither the center of gravity nor the center of oscillation; but will be at a great distance below the axis when p is great respect of b , as in the case of our experiments, in which p is 600 or 800 times b .

30. It

30. It may not be improper here, by the way, to enquire a little into the time of the penetration, its extent or depth, and the measure of the resisting force of the wood.

It was found above that

$$x = \frac{pgo u u}{4biiR}, \text{ and } x + z = \frac{b}{4bR} \times (v^2 - v'^2).$$

Now substituting in these $\frac{bii v}{pgo + bii}$, the greatest value of u , for u and v , we have

$$x = \frac{pgo}{4bR} \times \frac{b^2 i^2 v^2}{(pgo + bii)^2},$$

$$z = \frac{pgo}{4bR} \times \frac{b v^2}{pgo + bii}.$$

The latter of these being the greatest depth penetrated by the ball into the wood, and the former the distance moved by the point c of the pendulum at the instant when the penetration is completed. Both of which, it is evident, are directly as the square of the original velocity of the ball, and inversely as the resisting force of the wood; the other quantities remaining constant.

Hence also it appears that, other things remaining, the penetration will be less, as i is greater, or as the point of impact is farther below the axis. It is farther evident that the penetration will diminish as the sum of the forces pgo diminishes.

Now, for an example in numbers, a ball fired with a velocity of 1500 feet per second, has been found to penetrate about 14 inches into a block of sound dry elm, when the dimensions of the pendulum were as below:

$p = 660$ lb	the ball being cast iron,
$g = 78$ inches or $6\frac{1}{2}$ feet,	its diameter 1.96 inches,
$o = 84$ inches or 7 feet,	and its weight $1\frac{3}{4}$ or $\frac{67}{84}$ lb.
$i = 90$ inches or $7\frac{1}{2}$ feet,	and the value of z is 14 inch. or $\frac{7}{6}$ feet.

Here the value of v is 1500, and $z = 14$ inches or $\frac{7}{6}$ feet.

Hence $R = \frac{pgo}{4bz} \times \frac{bv^2}{pgo + bii} = \frac{660.13.7.6.67}{2.4.16.7.64} \times \frac{1500.1500}{660.6\frac{1}{2}.7 + \frac{67}{64} \cdot \frac{15}{2} \cdot \frac{15}{2}}$
 $= 32000$ nearly, which is the value of R for a ball of that size and weight. Or the resistance in this instance is 32000 times the force of gravity.

Hence also $x = \frac{pgo}{4bR} \times \frac{b^2 i^2 v^2}{(pgo + bii)^2} = \frac{1}{469}$ part of a foot, or $\frac{1}{39}$ part of an inch, is the space moved by the point c of the pendulum when the penetration is completed.

Also $\tau = \frac{pgo v}{2biiR} = \frac{2z}{v} = \frac{\frac{2}{3}}{1500} = \frac{1}{643}$ part of a second, is the time of completing the penetration of 14 inches deep.

31. Upon the whole then it appears, that our rule will give, without sensible error, the true velocity with which the ball strikes the pendulum. But this is not, however, the same velocity with which the ball issues from the mouth of the gun, which will indeed be something greater than the former, on account of the resistance of the air which the ball passes through in its way from the gun to the pendulum. And although this space of air be but small, and although the elastic fluid of the powder pursue and urge the ball for some distance without the mouth of the piece, and so in some degree counteract the resistance of the air, yet it will be proper to enquire into the effect of this resistance, as it will probably cause a difference between the velocity of the ball, as computed from the vibration of the pendulum and the vibration of the gun; which difference will, by the bye, be no bad way of measuring the resistance of the air, especially if the gun be placed at a good distance from the pendulum; for the vibration of the gun will measure the velocity with which the ball issues from the mouth of it; and the vibration of the pendulum the velocity with which it is struck by the ball.

32. To find therefore the resistance of the air against the ball in any case: it is first to be considered that the resistance to a plane moving perpendicularly

perpendicularly through a fluid at rest, is equal to the weight or pressure of a column of the fluid whose altitude is the height through which the body must fall by the force of gravity to acquire the velocity with which it moves through the fluid, the base of the column being equal to the plane. So that, if a denote the area of the plane, v the velocity, n the specific gravity of the fluid, and $b = 16.09$ feet; the altitude due to the velocity v being $\frac{vv}{4b}$, the whole resistance or motive force m will be

$$a \times n \times \frac{vv}{4b} = \frac{anvv}{4b}.$$

Now if d denote the diameter of the ball, and $k = .7854$, then shall $a = kd^2$ be a great circle of the ball; and consequently $m = \frac{knd^2v^2}{4b}$ = the motive force on the surface of a circle equal to a great circle of the ball.

But the resistance on the hemispherical surface of the ball is only one half of that on the flat circle of the same diameter; therefore $m = \frac{knd^2v^2}{8b}$ is the motive force on the ball; and if w denote its weight, $\frac{knd^2v^2}{8bw}$ will be equal to f the retarding force.

Since $\frac{2}{3}kd^3$ is the magnitude of the sphere, if N denote its density or specific gravity, its weight w will be $= \frac{2}{3}kd^3N$; consequently the retarding force f or $\frac{m}{w}$ will be $= \frac{knd^2v^2}{8b} \times \frac{3}{2kd^3N} = \frac{3nvv}{16dbN}$.

But by the laws of forces $v\dot{v} = 2bf\dot{x} = \frac{-3nvv}{8dN}\dot{x}$, and $\frac{\dot{v}}{v} = \frac{-3n}{8dN}\dot{x} = -e\dot{x}$, where x is the space passed over, putting $e = \frac{3n}{8dN}$, and making the value negative because the velocity v is decreasing. And the correct fluent of this is $\log. v - \log. v$ or $\log. \frac{v}{v}$ = ex , where v is the first or greatest velocity of projection. Or if A be $= 2.718281828$ &c. the number whose hyperbolic logarithm is 1,

then is $\frac{v}{v} = A^{ex}$, and hence the velocity $v = \frac{v}{A^{ex}} = v A^{-ex}$. So that the first velocity is to the last velocity, as A^{ex} to 1. And the velocity lost by the resistance of the medium is $(A^{ex} - 1) v$ or $\frac{A^{ex} - 1}{A^{ex}} v$.

33. Now to adapt this to the case of our balls, which weighed on a medium $16\frac{3}{4}$ ounces when the diameter was 1.96 inches; we shall have $1.96^3 \times .5236 =$ the magnitude of the ball; and as 1 cubic foot, or 1728 cubic inches, of water, weighs 1000 ounces; therefore $\frac{1728 \times 16\frac{3}{4}}{1000 \times 1.96^3 \times .5236} = 7.344 = N$ is the specific gravity of the iron ball; which is very justly something less than the usual specific gravity of solid cast iron, on account of the small air bubble which is in all cast metal balls. Also the mean specific gravity of air is .0012, which is the value of n . Hence $e = \frac{3n}{8dN} = \frac{3 \times .0012 \times 12}{8 \times 1.96 \times 7.344} = \frac{1}{2666}$.

Now the common distance of the face of the pendulum from the trunnions of all the guns, was $35\frac{1}{2}$ feet; and the distance of the muzzles of the four guns, was nearly $34\frac{1}{4}$ for the 1st or shortest gun, 34 for the 2d, 33 for the 3d, and $31\frac{1}{2}$ for the 4th. But as the elastic fluid pursues and urges the ball for a few feet after it is out of the gun, it may be supposed to counter-balance the resistance of the air for a few feet, the number of which cannot be certainly known, and therefore we shall suppose 32 feet to be the common distance, for each of the guns, which the ball passes through before it reach the pendulum. Hence then the distance $x = 32$; and consequently $ex = \frac{32}{2666} = \frac{16}{1333}$.

Then $A^{ex} - 1 = .01207 = \frac{1}{83}$ nearly. That is, the ball loses nearly the 83d part of its last velocity, or the 84th part of its first velocity, in passing from the gun to the pendulum, by the resistance of the air. Or the velocity at the mouth of the gun, is to the velocity at the pendulum, as 84 to 83; so that the greater diminished by its 84th part gives the less, and the less increased by its

83d part gives the greater. But if the resistance to such swift velocities as ours be about three times as great as that above, computed from the nature of perfect and infinitely compressed fluids, as Mr. Robins thinks he has found it to be, then shall the velocity at the gun lose its 28th part, and the greater velocity will be to the less, as 28 to 27. This however is a circumstance to be discovered from our experiments, or otherwise.

Of the Velocity of the Ball, as found from the Recoil of the Gun.

34. It has been said by more than one writer on this subject, that the effect of the inflamed power on the recoil of the gun, is the same whether it be charged with a ball, or fired by itself alone; that is, that the excess of the recoil when charged with a ball, over the recoil when fired without a ball, is exactly that which is due to the motion and resistance of the ball. And this they say they have found from repeated experiments. Now supposing those experiments to be accurate, and the deductions from them justly drawn; yet as they have been made only with small balls and small charges of powder, it may still be doubted whether the same law will hold good when applied to such cannon balls, and large charges of powder, as those used in our present experiments. Which is a circumstance that remains to be determined from the results of them. And this determination will be easily made, by comparing the velocity of the ball as computed from this law, with that which is computed from the vibration of the ballistic pendulum. For if the law hold good in such cases as these, then the velocity of the ball, as deduced from the vibration of the gun, will exceed that which is deduced from the vibration of the pendulum, by as much as the velocity is diminished by the resistance of the air between the gun and the pendulum.

35. Taking

35. Taking this for granted then in the mean time, namely, that the effect of the charge of powder on the recoil of the gun, is the same either with or without a ball, it will be proper here to investigate a formula for computing the velocity of the ball from the recoil of the gun. Now upon the foregoing principle, if the chord of vibration be found for any charge without a ball, and then for the same charge with a ball, the difference of those chords will be equal to the chord which is due to the motion of the ball. This follows from the property of a circle and a body descending along it, namely, that the velocity is always as the chord of the arc described in a semivibration.

Let then c denote this difference of the two chords, that is

c = the chord of arc due to the ball's velocity,

G = weight of the gun and iron stem, &c.

b = weight of the ball,

g = distance of center of gravity of G ,

o = distance of its center of oscillation,

n = its N°. of oscillations per minute,

i = distance of the gun's axis, or point of impact,

r = radius of arc or chord c ,

v = velocity of the ball,

V = velocity of the gun, or of the axis of its bore.

Then because biv is the sum of the momenta of the ball,
and GgV the sum of the momenta of the gun,
and because action and re-action are equal, these two must be equal to each other, that is

$$biv = GgV:$$

But because v is the velocity of the distance i , therefore by similar figures $i : o :: v : \frac{oV}{i}$ the velocity of the center of oscillation. And because the velocity of this center, is equal to the velocity generated by gravity, in descending perpendicularly through the height or versed sine of

of the arc described by it,

and because $2r : c :: c : \frac{cc}{2r} = \text{vers. sine to radius } r,$

and $r : o :: \frac{cc}{2r} : \frac{cco}{2rr} = \text{vers. sine to radius } o,$

therefore $\sqrt{b} : \sqrt{\frac{cco}{2rr}} :: 2b : \frac{c}{r} \sqrt{2bo}$, the velocity of the center of oscillation as deduced from the chord c of the arc described, where $b = 16.09$ feet; which velocity was before found $= \frac{ov}{i}$.

Therefore $\frac{ov}{i} = \frac{c}{r} \sqrt{2bo}$, or $ov = \frac{ci}{r} \sqrt{2bo}$.

Then this value of ov being substituted in the first equation $bii v = ggov$, we have $bii v = \frac{ggci}{r} \sqrt{2bo}$, and hence the velocity $v = \frac{ggc}{bir} \sqrt{2bo} = \frac{5.6727ggc}{bir} \sqrt{o}$, being the formula by which the velocity of the ball will be found in terms of the distance of the center of oscillation and the other quantities. Which is exactly similar to the formula for the same velocity, by means of the pendulum in Art. 22, using only g , or the weight of the gun, for $p + b$ or the sum of the weights of the ball and pendulum.

And if, instead of $\sqrt{o}$ be substituted its value $\sqrt{\frac{11737.5}{nn}}$ or $\frac{108.3398}{n}$, from Art. 20, it becomes $v = 614.58 \times \frac{ggc}{birn}$, or $= \frac{59000}{96} \times \frac{ggc}{birn}$, the formula for the velocity of the ball in terms of the number of vibrations which the gun will make in one minute, and the other quantities.

36. Farther, as the quantities g, b, i, r, n commonly remain the same, the velocity will be directly as the chord c . So that if we assume a case in which the chord shall be 1, and call its corresponding velocity u ; then shall $v = cu$; or the velocity corresponding to any other

other chord c , will be found by multiplying that chord c by the first velocity u answering to the chord 1.

Now, by the following experiments, the usual values of those literal quantities were as follows :

viz. $G = 917$

$g = 80.47$

$b = 1.047$, sometimes a little more or less.

$i = 89.15$

$r = 1000$

$n = 40.0$, for the gun n° 2, (but the 400th part more for n° 1, and the 400th part less for n° 3, and the 200th part less for n° 4.)

Then, writing these values in the theorem, instead of the letters, it becomes $v = 12.15c$. So that the number 12.15 multiplied by the difference between two chords described with any charge, the one with and the other without a ball, will give the velocity of the ball when the dimensions are as stated above. And when the values of any of the letters vary from these, it is but increasing or diminishing that product in the same proportion, according as the letter belongs to the numerator or denominator in the general formula $\frac{59000}{96} \times \frac{Ggc}{birn}$. When such variations happen, they will be mentioned in each day's experiments. And farther when only the values of G, g, i, n are as before specified, the same formula will become $12718 \times \frac{c}{br}$.

But note that these rules are adapted to the gun n° 2 only; therefore for n° 1 we must subtract the 400th part, and add the 400th part for n° 3, and add the 200th part for n° 4.



OF THE EXPERIMENTS.

37. WE shall now proceed to state the circumstances of the experiments for each day separately as they happened; by this means shewing all the processes for each set of experiments, with the failure or success of every trial and mode of operation; and from which also any person may recompute all the results, and otherwise combine and draw conclusions from them as occasion may require. Making but a very few cursory remarks on each day's experiments, to explain them when necessary; and reserving the chief philosophical deductions, to be drawn and stated together, after the close of the experiments, in a more connected and methodical way.

The machinery having been made as perfect as the circumstances would permit, 20 barrels of government powder were procured, all by the best maker, and numbered from 1 to 20. A great number of iron balls were also cast on purpose, very round, and their accidental asperities ground off: they were a little varied in their size and weight, but most of them almost equal to the diameter of the bore, so as to have but little windage. The powder was uniformly mixed, and every day exactly weighed off by the same careful man, and put up in very thin flannel bags, of a size just to fit the bore of the gun; a thread was tied round close by the powder, after being shaken down, and the flannel cut off close by the thread, so as to leave as short a neck as possible to the bag. The charge of powder was pushed gently down to the lower or breech end of the bore, and the same quantity of powder always made to occupy nearly the same extent, by means of the divisions of inches and tenths marked on the ramrod. The ball was then put in, without using any wads, and set close to the charge of powder, and kept in its place by a fine thread crossed two or three times about it, which by its friction gave it a hold of the sides of the bore, as the windage was very small. The gun was directed point blank, or horizontal, and

perpendicular to the face of the pendulum block, $35\frac{1}{2}$ feet distant from the trunnions, and was well wiped and cleaned out after each discharge, which was made by piercing the bottom of the charge through the vent, and firing it by means of a small tube. An account was kept of the barometer and thermometer, placed within a house adjoining, and shaded from the sun.

The machinery having been all prepared and set up in a convenient place in Woolwich Warren, Major Blomefield and I went out on the 6th of June 1783, with a sufficient party of men, to try the effects of them for the first time, which were as follows.

38. Friday, June 6, 1783; from 10 till 12 A. M.

The weather was warm, dry, and clear.

The barometer at 30.17, and thermometer at 60°.

The intention of this day's experiments, was to try and adjust the apparatus; to ascertain the proper distance of the pendulum; as also the comparative strength of the different barrels of powder, by firing several charges of it, without balls or wads. Out of the 20 barrels of powder, were selected the 6 which had been found to be most uniform, and nearest alike, by the different eprouvettes at Purfleet, which were n^{os} 2, 5, 13, 15, 18, 19; of which the first two only were tried this day, as below. The gun was the short one, n^o 1, and weighed this day, with leaden weights and iron stem, 906 lb: the distance of the tape, by which the chord of its recoil was measured, was not taken, and it was probably a little more than the usual length, 110 inches, employed in most of the experiments of this year.

Here

Here it appears that the quantity of recoil increased in a higher ratio than the quantity of powder.

The pendulum was not moved by the blast of the powder in these experiments.

No. of Experm.	Powder		Chord of recoil inches	Medium of recoil
	fort	weight oz		
1	n° 2	2	2.25	2.30
2		2	2.35	
3		2	2.30	
4	n° 5	2	2.55	2.50
5		2	2.40	
6		2	2.55	
7	n° 5	8	13.00	12.88
8		8	12.75	
9	n° 2	8	12.50	12.50
10		8	12.50	

39. *Saturday, June 7, 1783; from 9½ A. M. till 12.*

The weather cloudy or hazy, but it did not rain.

Barometer 30.25, Thermometer 60°.

To try all the 6 forts of powder, and the effect of the blast on the pendulum, when high charges are used.

The first 14 rounds were with the same apparatus and gun n° 1, as the former day.

The other four rounds with the gun n° 4, but without the leaden weights; it weighed with the iron 561 lb.

These recoils are very uniform, and there appears to be but little difference in the quality of the powder among the several forts.

All the charges were in flannel bags, except n° 14 and 18, of 16 oz each, for want of bags large enough provided to put it in. Each charge was rammed with two or three slight strokes. A considerable quantity of the powder of n° 14 was blown out unfired; many of the grains were found on the ground, and on the top of the pendulum block, and many were found sticking in the

N°.	Powder		Recoil	Mediums
	fort	weight oz	inches	
1	2	8	13.35	} 13.38
2		8	13.40	
3	5	8	13.50	} 13.28
4		8	13.05	
5	13	8	13.50	} 13.23
6		8	12.95	
7	15	8	13.50	} 13.35
8		8	13.20	
9	18	8	13.25	} 13.38
10		8	13.50	
11	19	8	12.95	} 12.95
12		8	12.95	
13	13	2	2.25	
14		16	26.00	
15	13	2	4.5	vibr. of pendulum 0
16		4	10.8	0
17		8	24.7	0.25
18		16	53.3	1.10

face of it. By the force of these striking it, and by the blast of the powder, or motion of the air, the pendulum was observed visibly to vibrate a little: but the measuring tape had not been put to it. This was therefore now added, to measure the vibration by. And, to try to what degree the pendulum would be affected by the explosion of the powder, the 7 feet amufette was suspended, and pointed opposite the center of the pendulum for the last 4 rounds. The pendulum was accordingly observed to move with the 8 ounces, but more with the 16 ounces, as appears at the bottom of the last column of the table above. The pendulum being thus much affected, we were convinced of the necessity of making a paper screen to place between the gun and the pendulum; which we accordingly did, and used it in the whole course of experiments, at least in the larger charges. At the last charge, which was 16 ounces of loose powder, much fewer grains were blown out than with the like charge at n° 14 with the short gun. The recoil at n° 14 is evidently less than it ought to be; owing

owing to the quantity of unfired powder that was blown out. It is remarkable that the recoil of the two guns, with the same charge, both for 2 ounces and 8 ounces, are nearly in the reciprocal ratio of the weights of the guns; a small excess only, over that proportion, taking place in favour of the long gun, as due to its superior length. The recoils are each visibly in a higher proportion than the charges of powder: for, in the last four experiments, the charges of 2, 4, 8, 16 ounces, are in the continued proportion of 1 to 2; while their recoils 4.5, 10.8, 24.7, 53.3, are all in a higher ratio than that of 1 to 2; for, dividing the 2d by the 1st, the 3d by the 2d, and the 4th by the 3d, the three successive quotients are 2.40, 2.29, 2.16, which are all above the double ratio, but approximating, however, towards it as the charge is increased. And farther, if we divide these quotients successively one by another, the two new ratios or quotients will be nearly equal. So that, ranging those recoils in a column under each other, and their two successive orders of ratios in the adjacent columns, we shall have in one view the law which they observe, as here below, where they always tend to equality.

$$\begin{array}{c|c|c|c} 4.5 & & & \\ 10.8 & 2.40 & & \\ 24.7 & 2.29 & .954 & \\ 53.3 & 2.16 & .944 & .99 \end{array}$$

Again, if we take successive differences between the same recoils, and between these differences, and then between the second differences, and so on, thus

$$\begin{array}{c|c|c|c} 4.5 & & & \\ 10.8 & 6.3 & & \\ 24.7 & 13.9 & 7.6 & \\ 53.3 & 28.6 & 14.7 & 7.1 \end{array}$$

the columns, as well as the lines, ascending obliquely from left to right, have their numbers approaching, and at length ending in the ratio of 2 to 1, the same as the quantities of powder.

40. *Friday, June 13, 1783; from 11 till 1 o'clock.*

The air moist, with small rain at intervals.

The gun n° 2 was mounted, and loaded with all the leaden weights: it was charged with the following quantities of powder; sometimes with a ball, and sometimes without one, as denoted by the cipher 0, in the columns of weight and diameter of ball. The radius to the tape was —. As these experiments were made only to discover if the leaden weights would render the gun sufficiently heavy, that the recoil might not be too large with the high charges of powder and ball, the pendulum block was removed, to let the balls enter and lodge in the bank which was behind it

Here again it appears that the recoils, without balls, are always in a greater ratio than the charges of powder. It also appears that the recoils, when balls are employed, are nearly in the ratio of the quantities of powder, when the charges are small; but gradually decreasing more and more below that ratio, as the charge of powder is increased. And if we subtract each recoil without a ball, from the corresponding recoil with a ball, for the same charge of powder, taking the differences as here below,

No.	Powder		Ball's			Recoil
	fort	wt	diameter	wt		
			inches	oz	dr	
1	19	2	0	0		2'5
2	19	2	0	0		2'5
3	19	4	0	0		5'2
4	19	8	0	0		13'5
5	19	16	0	0		28'1
6	18	16	0	0		28'0
7	19	2	1'9	15	4	8'9
8	19	4	1'9	15	4	16'15
9	19	8	1'9	15	4	26'5
10	19	16	1'9	15	4	41'75
11	18	16	1'9	15	4	34'3
12	18	16	1'9	15	4	35'15
13	19	16	1'975	16	13	36'0
14	19	16	1'965	16	13	33'5

Weight of powder	oz	2	4	8	16
Recoils with a ball		8.9	16.2	26.5	34.7
Recoils without -		2.5	5.2	13.5	28.0
Differences -		6.4	11.0	13.0	6.7

it will appear that those differences increafe as far as to the charge of 8 ounces, and then decrease again.

There must have been some mistake in the 10th round, as the recoil, which is 41.75 inches, is greater than can well be expected with that charge of powder. Probably the tape had entangled, and been drawn farther out in the return of the gun from the recoil.

41. *Monday, June 23, 1783.*

We went with the workmen, and took the weight and dimensions of the several parts of the machinery, both of the pendulum with its stem, and of the guns with their frame or iron stem, and the leaden weights to fit on about the trunnions.

OF THE PENDULUM.

Total weight with all the iron work about it	- - - - -	559 lb
Distance from its axis to the center of gravity	- - - - -	75.2 inches
Ditto - - - - to the tape or lowest point	- - - - -	115.1 inches
Ditto - - - - to the top of the block	- - - - -	76.3 inches
Dimensions of the wooden block	- - - - -	18, 22, and 24 inches
That is, breadth of the face 18, height of the face 24, and length from front to back 22.		

THE GUN FRAME OR STEM.

Total weight of all the iron work	- - - - -	188 lb
Distance from its axis to the center of gravity (without gun)	- - - - -	44.25 inches
Ditto - - - - to the tape or lowest point	- - - - -	110 inches
Ditto - - - - to center of the trunnions	- - - - -	90.3 inches
Ditto - - - - to the perpendicular arm	- - - - -	75.75 inches

The

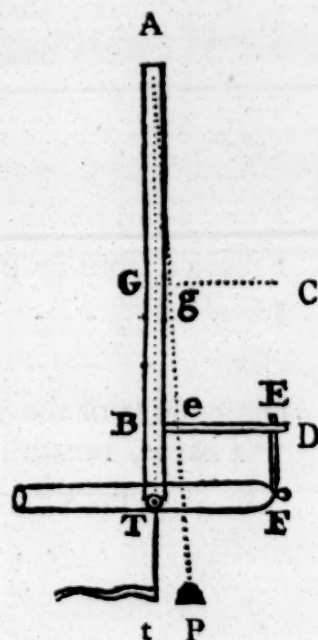
The following figure is a side-view of the gun-frame or stem, as it hung on its axis with the gun,

- A being the point through which the axis passes,
- G the point in the stem where it rests in equilibrio, shewing the distance AG of the center of gravity below the axis,
- Gg perpendicular to AG,
- AP a plumb-line cutting Gc in g,
- g the center of gravity of the iron work,
- BD a fixed perpendicular arm,
- EF a sliding piece to support the gun,
- T the center of the trunnions,
- t the place of the tape or lowest point.

And the dimensions or measures to these points are as follows:

		inches
AG	- -	44'25
AB	- -	75'75
AT	- -	90'3
At	- -	110'0
Be	- -	5'6
Gg	- -	3'3

Breadth of stem AT - - 3'5, and from the middle of this breadth the distances Be and Gg are measured.



42. The following are also the measures taken to settle the position of the compound center of gravity of the gun with its leaden weights and iron stem all together.

N ^o of the gun	Diameter of the trunnions	Diameter of the gun at the center of the trunnions	Center of gravity or axis of the gun alone				Center of gravity of the whole below axis
			behind the		above cent. of the trun.	below axis of vibration	
			muzzle	center of trunnions			
1	2.2	7.00	18.5	1.4	1.24	89.06	80.47
2	2.2	5.89	24.5	1.8	1.24	89.06	80.47
3	2.25	5.06	37.4	4.2	1.11	89.19	80.50
4	2.2	4.84	51.3	3.0	1.02	89.28	80.44

The numbers in the last column of this table, are the values of the letter g , in the formula for the velocity by means of the recoil of the gun. This letter may always be supposed to have the value 80.47 inches, as the two last numbers of the column differ from it but .03 only, or about the 2700 part of the whole, inducing an error of only about half a foot in the velocity of the ball.

The values of g , in this last column of the table, were computed in the following manner.

Suppose the centers of gravity of the stem, gun, and leads to be all reduced to the line of the stem $A T$ connecting the axis and center of the trunnions T , in which they are situated, very nearly;

so that G be the center of gravity of the stem,

C - - - - - of the gun,

T - - - - - of the leads;

also the numbers on the right-hand side of these points, namely, 44.25 and 89.06 and 90.3, are the measured distances below the axis at A ; and the numbers on the left-hand side, namely, 188, 290, and 439, are the weights of the bodies belonging to those centers of gravity. Then, from the property of the center of gravity, we shall have these operations

	A	
188	G	44.25
917	E	80.47
290	C	89.06
729	D	89.81
439	T	90.30

439	90.30	
290	89.06	
729 : 290 ::	1.24 :	0.49 - - T D
		90.30 - - A T
		89.81 - - A D
		44.25 - - A G
729		
188		45.56 - - G D
917 : 729 ::	45.56 :	36.22 - - G E
		44.25 - - A G
		80.47 - - A E

where D is the center of gravity of the bodies at the points C and T , or of the gun and leads; and E the center of gravity of the two bodies placed at the points G and D , or of all the three bodies at the points G , C , T ; that is, E is the compound center of gravity for both gun, iron, and

and leads in one mafs. And the some operation is to be repeated for the other guns.

43. It may here be alfo remarked, that the mean number of vibrations per minute, for every gun, weighing in all 917lb, taken among the actual vibrations of each day, is for

n ^o 1	n ^o 2	n ^o 3	n ^o 4
40.1	40.0	39.9	39.8

which number must be used as the true value of n , in the formula for the velocity of the ball by means of the recoil of the gun. The number of the gun's vibrations was commonly tried every day, and they were found to vary but little, and among them all the numbers above-mentioned, are the arithmetical mediums.

44. Moreover the mean numbers for the pendulum, among all the daily measurements of its weight, center of gravity, and oscillations per minute, are thus :

weight	g	n
660lb	77.3	40.2

Of the great number of these measures that were taken, the variations among them would be sometimes in excess and sometimes in defect; and therefore the above numbers, which are the means among the whole, as long as the iron work remains the same, will probably be very near the truth. And by using always these, with proportional alterations in g and n for any alteration in the weight p , the computations of the velocity of the ball will be made by a rule that is uniform, and not subject at least to accidental single errors. When the weight of the pendulum varies by the wood alone of the block, or the straps about it, the alteration is to be made at the center of the block, which is exactly 88.3 inches below the axis; that is, in that case the value of i is 88.3 in the formula

$\frac{i-g}{p+b}$, or the correction for g ; and in $\frac{bin(inn-140850)}{140850(2pg+bi)+biinn}$, the correction

rection of n . But when the alteration of the weight p arises from the balls and plugs lodged in the same block, then the value of i in those corrections is the medium among the distances of the point struck. And when the iron work is altered, the middle of the place altered gives the value of i in the same theorems.

In these corrections too p denotes 660, g 77.3, n 40.2, and b the difference between 660 and any other given weight of the pendulum; which value of b will be negative when this weight is below 660, otherwise positive; so that $p + b$ is always equal to this weight of pendulum.

And if these values of p , g , n be substituted for them in those corrections, they will become

$$\frac{i - 77.3}{660 + b} b \text{ or } \frac{i - 77.3}{p} b, \text{ the correction for } g, \text{ and}$$

$$\frac{40.2bi(i - 86.3)}{8805707 + bi(i + 86.3)} \text{ the correction for } n.$$

And farther, when $i = 88.3$, the same become

$$\frac{11b}{660 + b} \text{ or } 11 - \frac{7260}{p} \text{ the correction for } g, \text{ and}$$

$$\frac{b}{1263 + 2.2b} \text{ or } .4545 - \frac{261}{p - 86} \text{ the correction for } n,$$

as adapted to an alteration at the center of the pendulum.

And in that case

$$G = 88.3 - \frac{7260}{p} \text{ is the new value of } g, \text{ and}$$

$$N = 39.7454 + \frac{261}{p - 86} \text{ is the new value of } n.$$

But those corrections will have contrary signs when b is negative, as well as the second term in each of the denominators.

45. Monday, June 30, 1783; from 9½ A. M. to 2½ P. M.

The air clear, dry, and hot.

Barometer 30.34, and Thermometer 74.

We began this day for the first time to fire with balls against the pendulum block. The powder of the six barrels before-mentioned, had been all well mixed together for the use of our experiments, that they might be as uniform as possible, in that, as well as in other respects.

The GUN was n° 1, with the leaden weights.

Its weight and the distance of its center of gravity, were as before-mentioned; the distance of the tape it was forgotten this day to measure, but from circumstances judged to be 106½.

PENDULUM. Its weight - - - - - 559 = *p*

Distance to the tape - - - 115.1 = *r*

Nº	Powder	Ball's			Vibration of		Struck below axis	Plugs	Values of			Veloc. of the ball
		wt		diam.	gun	pend.			<i>p</i>	<i>g</i>	<i>n</i>	
	oz	oz	dr	inches	inches	inches	inches	inch	lb	inches		feet
1	2				2							
2	2				2							
3	8				11.2							
4	16				23.4							
5	16	16	13	1.95	34	23	87.9		559.0	75.30	40.30	1392
6	16	16	13	1.95	35.4	25	86.8		560.1	75.32	40.30	1534
7	16	16	13	1.95	34.8	23.7	88.8		561.1	75.35	40.29	1426
8	16	16	13	1.95	35.1	25	87.6	15	562.2	75.37	40.29	1530
9	16	16	13	1.95	35.7	23.7	88.2	10	564.6	75.39	40.28	1445
10	16	16	13	1.95	35.2	23.1	88.3		566.5	75.42	40.28	1412
medium					35.0					medium		1456

The first 4 rounds were with powder only; the other 6 with balls, all of the same size and weight.

The diameter of the gun bore being 2.02, and
the diameter of the ball - - - - 1.95, consequently

the windage was - - - - - 0.07

Mean length of the charge of powder 10.6

The two oaken plugs which were driven in, to fill up the holes, after the 8th and 9th rounds, weighed about $1\frac{1}{4}$ oz. to each inch of their length. The whole weights of these plugs, and the weights of the balls lodged in the block, were continually added to the weight of the pendulum, to compleat the numbers for the values of p in the 9th column; and from these numbers the correspondent values of g and n , in the next two columns, are computed by their proper corrections in Art. 23 and 25. After which the velocities contained in the last column are computed by the formula in Art. 24. And the medium among all these velocities, as well as that of the recoils of the gun, are placed at the bottom of their respective columns.

From the mean recoil with ball 35.0

take the recoil without a ball 23.4

there remains - - - - $c = 11.6$

Then, having $b = 1.051$, and $r = 106.5$, by the rule $12718 \times \frac{c}{br}$ in Art. 36, we have only 1315 feet, for the velocity of the ball as deduced from the recoil of the gun; which is 141 less than the velocity found by the vibration of the pendulum, or about $\frac{1}{10}$ th of the whole velocity.

The powder blown out unfired was not much. The apparatus performed all very well, except only that the wood of the pendulum seemed not to be very sound, as it was pierced quite through by the end of this day's experiments; though the sheet lead with which the back was covered, as well as the face, just prevented the balls and pieces of the wood from falling out at the back of the pendulum.

46. Saturday,

46. *Saturday, July 5, 1783; from 9 till 2 o'clock.*

The weather clear, dry, and hot.

Barometer 30.27, and Thermometer 74.

G U N, n° 3.

Weight - - - - - 917
To center of gravity - - 80.47
To the tape - - - - - 109.7

P E N D U L U M.

Weight - - - - - 846
To center of gravity - - 79.6
To the tape - - - - - 117.3

N°	Powder	Ball's		Chord of vibrat.		Point struck	Plugs	Values of			Veloc. of the ball
		wt	diam	gun	pend			p	g	n	
	oz	oz	dr	inches	inches	inches		lbs	inches		
1	2			2.3							
2	2			2.3							
3	8			13.0							
4	8			14.1							
5	8			13.6							
6	16			26.3							
7	16			28.7	9.5						
8	16			26.5	0.3						
9	16	16	13	1.95	39.0	24.2	89.0	846.0	79.3	41.48	

A large piece had been cut out of the middle part of the pendulum, from the face almost to the back, to clear away the damaged part of the wood; and the vacuity was run full of lead, from an idea that the pendulum would not so soon be spoiled, and consequently that it would need less repairs. But this did not succeed at all; for the only shot we discharged, namely, n° 9, would not lodge in the lead, but broke into a thousand small pieces, many of which stuck in the lead, and formed a curious appearance; but the greater number rebounded back again, to the great danger of the by-standers. The ball made a large round excavation in the face of the lead, of 5 inches diameter in the front, and $3\frac{1}{2}$ inches deep in the center of the hole.

Length of the charge of 16 oz was 11 inches.

47. *Friday,*

47. Friday, July 11, 1783; from 9 A. M. till - - - .

Fine, clear, hot weather.

G U N, n° 3.

Weight - - - - 917
To center of gravity - - 80.47
To the tape - - - - 110

P E N D U L U M.

Weight - - - - 610
To center of gravity - - 76.4
To the tape - - - - 118

N°	Powder	Ball's		Chord of vibrat.		Point struck	Plugs	Values of			Veloc. of the ball
		wt	diam	gun	pend			p	g	n	
	oz	oz	dr	inches	inches	inches					
1	2			2.5							
2	2			2.5							
3	16			28.4							
4	16			25.7							
5	16			28.3							
6	16	16	13	1.95	44.6	34.0	89.1				

Length of the charge of 16 oz was 11.2 inches.

The pendulum had been altered since the former day. The core of lead being taken out, some layers of rope were laid at the bottom of the hole, then the remainder up to the front filled with a piece of sound elm, and the face covered with sheet lead.

At the last round, or that with ball, the iron tongue which held the tape of the pendulum, having slipped down by the loosening of a screw, was strained and bent. Which stopped the experiments till it could be repaired.

48. Saturday,

48. Saturday, July 12, 1783; from 9 A. M. till - - - .

Fine, clear, hot weather.

The pendulum, gun n^o 3, and apparatus, were in every respect the same as in the last day's experiments, excepting that the radius of the tape, in the gun, was 110.2 inches instead of 110.

N ^o	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		wt		diam	gun	pend			<i>p</i>	<i>g</i>	<i>n</i>	
	oz	oz	dr	inches	inches	inches	inches		lb	inches		feet
1	2				2.5							
2	2				2.5							
3	16				28.0							
4	16				29.0							
5	16				28.6							
6	16	16	13	1.96	44.1	33.7	89.6		607.0	76.34	40.25	2151
7	16	16	13	1.96	42.6	30.9	90.3		608.1	76.36	40.25	1960
8	16	16	13	1.96	46.8	32.3	89.3		609.1	76.38	40.24	2076
9	16	16	13	1.96	44.4	30.5	89.6		610.2	76.39	40.24	1958
10	16	16	13	1.96	43.9	31.4	89.2		611.2	76.41	40.24	2028
11	16	16	13	1.96	42.3	31.5	90.7		612.3	76.43	40.23	2005
medium					44.0						medium	2030

The mean length of the charge of 16 oz was 11.7 inches. But this height was always taken when the cartridge was uncompressed: so that the powder lay looser than in former experiments. By a small pressure it occupied about $\frac{1}{4}$ of an inch less space.

The value of *p* at beginning this day is made a little less than the pendulum weighed at first, for reasons to be mentioned hereafter.

The mean recoil with a ball is 44.0, and without a ball 28.5, the difference of which is 15.5 = *c*. Also, in the formula for the velocity by means of the gun, we have *b* = 1.051, and *r* = 110.2. Consequently $v = \frac{401}{400} \times 12718 \times \frac{c}{br} = 1706$ for the velocity by that method. But the mean velocity by the pendulum is 2030, which exceeds the former by 324, or almost $\frac{1}{6}$ of the whole velocity.

49. *Thursday, July 17, 1783; from 12 till 3 P. M.*

Fine, clear, hot weather.

Barometer 30.23, Thermometer 72° at 9 o'clock.

G U N, n° 1.

Weight - - - - - 917
To center of gravity - - 80.47
To the tape - - - - - 110.2

It swung very freely, and would have continued its vibrations a long time; owing to the ends of the axis being made to turn or roll upon a convex iron support, and kept from going backward and forward, with the vibrations, by two upright iron pins, placed so as not quite to touch the axis, but at a very small or hair-breadth distance from it.

P E N D U L U M.

Weight - - - - - 657
To center of gravity - - 77.26
To the tape - - - - - 118

The pendulum would not vibrate longer than 1 minute before the arcs became imperceptible, owing to the friction of the upright pins, which touched and bore hard against the sides of the axis, unlike those of the gun, although they had the same kind of round support to roll upon. The pendulum had been well repaired, and strengthened with iron bars, and straps going round it in several places, except over the face. Also thick iron plates were let into, and across it, near the back part, then over them was laid a firm covering of rope, after which the rest of the hole was filled up with a block of elm, and finally the face covered over with sheet lead.

N ^o	Powder	Ball's		Vibration of		Point struck	Plugs	Values of			Veloc. ball	
		wt	diam	gun	pend			p	g	n		
	oz	oz	dr	inches	inches	inches	inch	lb	inches		feet	
1	2			2.3								
2	2			2.4								
3	8			12.3								
4	8			13.8								
5	8			11.9								
6	8			13.0								
7	8			13.2								
8	8	16	13	1.96	26.3	20.5	88.6	10	657.0	77.26	40.20	1450
9	8	16	13	1.96	27.2	21.9	89.7	9	658.5	77.28	40.20	1534
10	8	16	13	1.96	25.9	20.3	89.9	9	660.1	77.30	40.20	1423
11	8	16	13	1.96	26.8	20.5	88.6	8	661.6	77.33	40.20	1462
12	8	16	13	1.96	26.3	20.4	88.5	8	663.2	77.35	40.20	1460
13	8	16	13	1.96	26.8	20.9	88.6	8	664.7	77.37	40.20	1497
											mean	1471

The mean length of the charge of 8 oz was 5.9.

The pendulum, having been so well secured, suffered but little by this day's firing, only bulging or swelling out a little at the back part. All the balls were left in it, and all the holes were successively plugged up with oaken pins of near 2 inches diameter, which weighed 11 oz to every 10 inches in length.—The arcs described, both by the gun and pendulum, are pretty regular. And the whole forms a good set of experiments.

The mean recoil of the gun with ball 26.55
without ball 12.84

difference $c = 13.71$

Then $v = \frac{199}{200} \times 12718 \times \frac{c}{br} = \frac{399}{400} \times 12718 \times \frac{13.71}{1.051 \times 110.2} = 1501$,

the velocity of the ball as deduced from the recoil of the gun; which exceeds that deduced from the pendulum by 30, or nearly $\frac{1}{49}$ th part of this latter.

X 2

50. Friday,



50. Friday, July 18, 1783, from 9 A. M. till 12.

Fine and warm weather.

Barometer 30.28, and Thermometer 68° at 9 A. M.

N ^o	Powder	Ball's		diam inches	Vibration of		Point struck inches	Plugs inch	Values of			Veloc. ball feet
		wt oz	dr		gun inches	pend inches			p lb	g inches	n	
1	2				2.5							
2	2				2.55							
3	4				6.45							
4	4				6.05							
5	8				13.8							
6	8				13.9							
7	8				13.55							
8	8	16	13	1.96	28.35	24.35	87.8	9	664.7	77.37	40.19	1764
9	8	16	13	1.96	28.7	24.35	88.0	8	666.3	77.40	40.19	1765
10	8	16	13	1.96	28.3	24.3	87.9	7	667.8	77.42	40.19	1768
11	4	16	13	1.96	18.3	18.9	87.8	6	669.4	77.44	40.19	1380
12	4	16	13	1.96	18.0	18.4	87.3	6	670.9	77.47	40.19	1352
13	4	16	13	1.96	16.7	16.8	87.8	5	672.5	77.49	40.19	
14	4	16	13	1.96	18.4	18.0	87.7	4	674.0	77.51	40.19	1327

A fresh barrel of the mixed powder was opened for use this morning; and in the first 7 rounds, which were with powder only, that of the old and new barrel were used alternately, but no difference was observed.—The length of the charge of 4 oz was 3.2, and that of 8 oz was 5.9 inches.

The GUN was n^o 3.—Its weight 917

To center of gravity 80.47

To the tape - - - 110

It fwung so freely, that after many hundred vibrations the arcs were scarce fenfibly diminished. This gun heated more at the muzzle than n^o 1 did, being much thinner in metal there : but it was never very hot

to the hand in that part, and very little indeed about the place of the charge; for the heat was gradually less and less all the way from the muzzle to the breech, where it was not sensible to the hand.

The PENDULUM. Its weight at first round 664.7

To the tape - - - - 117.8

It had remained hanging since the last day's experiments, with all the balls and plugs in it, which increased its weight by 10 lb, except an allowance for evaporation, and increased the distance of the center of gravity by little more than $\frac{1}{10}$ th of an inch. It vibrated with great freedom; for it had this day been made to turn very freely on its axis, by placing the upright pins, which confine it side-ways, so as not quite to touch the axis, like those of the gun yesterday; and the effect was very great indeed, for it appeared as if it would have vibrated for a great length of time; whereas on the former days it stopped motion in about 1 minute, or at least after that the arcs soon became too small to be counted.—By this day's firing the pendulum seemed not to be much injured, the back part not appearing to be altered, and the fore part only a little swelled out, the piece of wood, that had been fitted in there, starting a little forward, and bulging out the facing of lead.

Of the plugs every 10 inches in length weighed 11 oz

	4 oz	8 oz
The mean recoil of gun with ball	18.23 -	28.45
without	6.25 -	13.75

the difference or $c = 11.98 - 14.70$

Hence the velocity by the recoil is 1321 - 1620

Mean ditto by the pendulum - - 1353 - 1766

Which exceeds that by recoil by 32 - 146

Or the 42d - 12th part.

This appears to be a good set, being very uniform, except the 13th round, which has been omitted, as evidently defective in the arc described both by the gun and pendulum, from some undiscovered and unaccountable cause.

51. Saturday,

51. Saturday, July 19, 1783; from 9 till 3.

A fine and warm day.

Barometer 30.12, Thermometer 70° at 9 o'clock.

N ^o	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		wt		diam	gun	pend			p	g	n	
	oz	oz	dr	inches	inches	inches	inches	inch	lb	inches		feet
1	2				2.3							
2	2				2.4							
3	4				5.8							
4	4				5.8							
5	4				5.8							
6	4	16	13	1.96	15.9	14.8	89.8	5	674.0	77.51	40.19	1065
7	4	16	13	1.96	16.3	15.4	89.8	5	675.3	77.54	40.19	1111
8	4	16	13	1.96	16.4	15.8	90.2	5	676.6	77.56	40.19	1137
9	4	16	13	1.96	16.3	15.4	89.3	5	677.9	77.58	40.19	1122
10	2	16	13	1.96	9.8	10.7	89.1	3	679.2	77.61	40.19	783
11	2	16	13	1.96	9.9	11.0	89.5	2	680.5	77.63	40.18	803
12	2	16	13	1.96	10.1	11.1	90.3	3	681.8	77.66	40.18	805
13	2				2.55							
14	2				2.55							
15	2	16	13½	1.96	11.1	12.6	88.8	3	683.1	77.68	40.18	930
16	2	16	13½	1.96	10.7	12.0	89.4	3	684.4	77.71	40.18	881
17	2	16	12½	1.96	11.0	12.3	89.8	3	685.7	77.73	40.18	901
18	2	16	12½	1.96	10.8	11.9	89.0		687.0	77.76	40.18	881

Of the plugs every 10 inches weighed 11 ounces.

Length of the charge of 2 oz was 1.7; and that of 4 oz was 3.2.

The GUN was n^o 1 for the first 12 n^os, and n^o 3 for the rest; in order to complete the comparison between these two guns with 2, 4, 8, and

16 oz of powder. The radius to the tape 110 inches, and the other circumstances as before.

The PENDULUM had been left hanging since yesterday, and the radius to the tape was 117.8 as before. It became however so full of balls and plugs to-day, that no more plugs could be driven in, all the iron straps being bent and forced out to their utmost stretch. It was therefore ordered to be gutted and repaired.

This is a good set of experiments; all the apparatus having performed well; and the arcs described, both by the gun and pendulum, being very uniform.

	Gun 1		Gun 3
	2 oz	4 oz	2 oz
Mean recoil with ball -	9.93	16.23	10.90
Ditto without - - -	2.35	5.80	2.55
The difference or $c =$	7.58	10.43	8.35
Hence velocity by recoil	832	1145	921
Mean ditto by pendulum	797	1109	898
Which are below recoil	35	36	23
Or nearly the part - -	$\frac{1}{23}$	$\frac{1}{31}$	$\frac{1}{38}$

52. Wednesday,

52. *Wednesday, July 23, 1783; from 10 till 3.*

Fine weather.

Barometer 29.85, Thermometer 70° at 4 P. M.

N ^o	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		wt		diam	gun	pend			p	g	n	
	oz	oz	dr	inches	inches	inches	inches	inch	lb	inches		feet
1	2				2.5							
2	2				2.5							
3	4				5.7							
4	4				5.7							
5	8				13.0							
6	8				12.9							
7	16				25.9							
8	16				26.6							
9	16				24.9							
10	16				26.5							
11	16	16	13½	1.96	39.9	22.4	87.7	12	690.0	77.78	40.18	1693
12	16	16	12½	1.96	39.0	21.3	88.3	12	691.6	77.80	40.18	1610
13	16	16	13½	1.96	41.2	23.1	88.7	9	693.2	77.82	40.18	1736
14	16	16	12½	1.96	38.3	21.2	88.7	9	694.9	77.85	40.18	1603
15	8	16	13½	1.96	27.9	21.4	89.2	8	696.5	77.88	40.18	1607
16	8	16	12½	1.96	27.1	20.2	88.6	7	698.1	77.90	40.17	1538
17	8	16	13½	1.96	27.3	20.5	88.9	8	699.8	77.92	40.17	1553
18	4	16	13½	1.96	16.8	15.1	88.0	7	701.4	77.94	40.17	1159
19	4	16	13½	1.96	16.7	14.7	88.3	6	703.0	77.96	40.17	1127
20	4	16	13½	1.96	16.6	14.6	88.5	5	704.7	77.99	40.17	1120
21	2	16	13½	1.96	9.5	9.4	88.6	4	706.3	78.01	40.16	722
22	2	16	13½	1.96	10.7	10.9	87.8	6	707.9	78.03	40.16	847
23	2	16	13½	1.96	9.6	9.4	88.5	5	709.6	78.05	40.16	727
24	2	16	13½	1.96	10.8	11.2	87.7		711.2	78.08	40.16	878

Length of charge of 2 oz was 2.1 inches

4 - - 3.3

8 - - 6.1

16 - - 10.9

Of

Of the plugs every 10 inches weighed 12 ounces.

The GUN was n° 2.—Its weight - - - 917
 To center of gravity 80.47
 To the tape - - - 110
 Oscillation per min. 40.6, as before.
 It heated very little by firing.

The PENDULUM.—Its weight - 690
 To the tape 117.8

It had been gutted, and repaired, by placing a stratum of lead, of 2 inches thick, before the iron plate, then the lead was covered with a block of wood, and the whole faced with sheet lead.

	2 oz	4 oz	8 oz	16 oz
Mean recoil with ball -	10.15	16.7	27.43	39.60
Ditto without - - -	2.50	5.7	12.95	25.98
The difference or $c =$	7.65	11.0	14.48	13.62
Hence velocity by recoil	840	1207	1592	1499
Mean ditto by pendulum	793	1135	1566	1660
Difference - - -	+ 47	+ 72	+ 26	- 161
Or the part - - -	$\frac{1}{17}$	$\frac{1}{16}$	$\frac{1}{60}$	$\frac{1}{10}$

So that the recoil gives the velocity with 2, 4 and 8 ounces of powder greater, but with 16 ounces much less, than the velocity shewn by the pendulum.

53. *Monday, July 28, 1783; from 10 till 2.*

A very hot day.

Barometer 29.74; Thermometer 77° at 10 A. M.

N ^o	Powder	Ball's		Vibration of		Point struck	Plugs	Values of			Veloc. ball
		wt	diam	gun	pend			<i>p</i>	<i>g</i>	<i>n</i>	
	oz			inches							
1	2			2.6							
2	2			2.45							
3	2			2.4							
4	2			2.45							
5	2			2.7							
6	2			2.65							
7	2			2.6							
8	4			6.3							
9	4			6.35							
10	4			6.35							
11	8			13.8							
12	8			14.0							
13	8			14.1							
14	16			28.1							
15	16			27.9							

	2 oz	4 oz	8 oz	16 oz
Mean length of charge	1.9	3.3	6.2	11.0
Mean recoil of gun -	2.65	6.33	13.97	28.0
Ditto with greater wt -	2.48			

The GUN n^o 4.—Its weight in first 4 rounds 1003
 Ditto in all the rest - - 917
 Other circumstances as before.

The

The gun was very hot before firing, with the heat of the sun. But heated little more with firing. It was hottest at the muzzle, where the hand could not long bear the heat of it.

The PENDULUM had been gutted and repaired since the last day.

It weighed - - 702

To the tape - - 117.8

No balls were fired this day.

54. Tuesday, July 29, 1783; from 12 till 3.

A fine and warm day.

Barometer 29.90; Thermometer 72° at 10 A. M.

N ^o	Powder	Ball's		Vibration of		Point struck	Plugs	Values of			Veloc. ball	
		wt	diam	gun	pend			p	g	n		
	oz	oz	dr	inches	inches	inches	inch	lb	inches		feet	
1	2			3.0								
2	2			2.7								
3	2			2.8								
4	2			2.75								
5	4			6.45								
6	4			6.25								
7	4			6.35								
8	8			14.4								
9	8			14.3								
10	8			14.5								
11	16			29.15								
12	16			28.25								
13	16			29.2								
14	16			28.3								
15	8	16	13	1.96	29.6	25.8	89.5	12	700.0	77.92	40.17	1946
16	8	16	13	1.96	29.1	25.0	89.0	11	702.0	77.95	40.17	1902
17	8	16	13	1.96	29.1	25.8	89.5	10	704.0	77.98	40.17	1959
18	16	16	13	1.96	44.5	29.1	89.4	13	706.0	78.01	40.16	2219
19	16	16	13	1.96	43.0	27.0	89.7	9	708.0	78.03	40.16	2058
20	16	16	12½	1.96	43.6	28.8	89.7	9	710.0	78.05	40.16	2207

Of the plugs every 10 inches weighed $13\frac{1}{2}$ ounces.

The GUN n° 4.—Its weight and other circumstances as usual. It did not become near so hot as yesterday.

The PENDULUM was as weighed and measured yesterday, having hung unused.

The tape drawn out in the last three rounds, both of the gun and pendulum, was rather doubtful, owing to the wind blowing and entangling it.

	2 oz	4 oz	8 oz	16 oz
Mean length of charge -	1.8	3.4	5.6	10.8
Mean recoil with ball -	.	.	29.27	43.70
Ditto without - - -	2.81	6.35	14.40	28.72
Difference or $c =$ -	.	.	14.87	14.98
Hence velocity by recoil	.	.	1643	1656
Mean ditto by pendulum	.	.	1936	2161
Difference, very great, -	.	.	293	505
Or the part - - - -	.	.	$\frac{1}{7}$	$\frac{1}{4}$

55. Wednesday,

55. *Wednesday, July 30, 1783; from 10 till 12.*

A fine day, moderately warm.

Barometer 30.06; Thermometer 69° at 12 o'clock.

N ^o	Powder	Ball's		Vibration of		Point struck	Plugs	Values of			Veloc. ball
		wt	diam	gun	pend			<i>p</i>	<i>g</i>	<i>n</i>	
	oz	oz	dr	inches	inches	inches	inch	lb	inches		feet
1	2			2.7							
2	2			2.6							
3	4			6.2							
4	4			6.0							
5	4	16	13	1.96	17.8	88.7	8	709.8	78.04	40.16	1376
6	4	16	13	1.96	17.8	86.7	9	711.3	78.07	40.16	1380
7	4	16	13	1.96	17.75	87.2	10	712.9	78.10	40.16	1368
8	2	16	13	1.96	11.0	87.8	7	714.4	78.12	40.15	960
9	2	16	12	1.96	11.25	86.8	7	715.9	78.15	40.15	993
10	2	16	12	1.96	10.9	87.3	5	717.5	78.18	40.15	951

The GUN was again n^o 4, and every circumstance about it as before.

The PENDULUM the same as left hanging since yesterday, with the addition of the balls and plugs in it.

This day's experiments a good set.

	2 OZ	4 OZ
Mean length of charge - -	1.7	3.24
Mean recoil with ball - -	11.05	17.78
Ditto without - - - -	2.65	6.10
Difference, or <i>c</i> - - -	8.40	11.68
Hence velocity by the recoil	929	1295
Mean ditto by the pendulum	968	1375
Difference, gun less - -	39	80
Or the part - - - -	$\frac{1}{14}$	$\frac{1}{17}$

56. *Thursday,*

56. *Thursday, July 31, 1783; from 10 till 12.*

Fine warm weather.

Barometer 30.3; Thermometer 69° at 10 A. M.

N ^o	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		wt		diam	gun	pend			<i>p</i>	<i>g</i>	<i>n</i>	
	oz	oz	dr	inches	inches	inches	inches	inch	lb	inches		feet
1	2				2.5							
2	16				23.8							
3	16				25.9							
4	16				23.8							
5	16				23.5							
6	16	16	13	1.96	37.3	17.8	89.6	6	717.2	78.18	40.15	1379
7	16	16	13	1.96	37.3	18.9	90.5	6	718.6	78.20	40.15	1453
8	16	16	13	1.96	34.5	16.4	90.2	6	720.1	78.22	40.15	1268
9	12	16	13	1.96	31.7	17.7	89.2	5	721.6	78.24	40.15	1387
10	12	16	12½	1.96	33.2	18.9	89.8	8	723.0	78.26	40.14	1477
11	12	16	12½	1.96	30.8	17.5	89.8	8	724.5	78.28	40.14	1371
12	12				21.0							
13	12				18.3							
14	12				18.8							

The GUN n^o 1.—Weight and every thing else as usual.—The annular leaden weights, which fit on about the trunnions, have gradually been knocked much out of form by the shocks of the sudden recoils; so that, not fitting closely, they are subject to shake, a circumstance which probably has occasioned the irregularities in the recoils of this day.

The PENDULUM continued hanging still. It is suspected that its vibrations are not to be strictly depended on with the high charges of powder; owing to the striking of the balls against the iron plate within the block, and so perhaps causing them to rebound within it, and disturb

turb the vibrations, which are not regular this day. After it was taken down, the pendulum was found to weigh 726 lb. But, from the weight of the balls and plugs lodged in it, it ought to have weighed 732 lb. It is therefore likely that the 6 lb had been lost, by evaporation of the moisture, in the 4 days, which is $1\frac{1}{2}$ lb per day. At the beginning of each day's experiments therefore $1\frac{1}{4}$ lb is deducted from the weight of the pendulum, or 2 lb before each of the last three days. And the like was done on some former days, for the same reason, when it appeared necessary.

Of the plugs, 10 inches weighed 10 ounces.

	12 OZ	16 OZ
Mean length of the charge -	8.4	11.1
Mean recoil with ball - -	31.9	36.4
Ditto without - - - -	19.4	24.25
Difference, or $c =$ - -	<u>12.5</u>	<u>12.15</u>
Hence velocity by the recoil	1374	1334
Mean ditto by the pendulum	<u>1412</u>	<u>1367</u>
Difference, the gun less -	38	33
Or nearly the part - - -	$\frac{1}{37}$	$\frac{1}{41}$

57. Tuesday,

57. *Tuesday, August 12, 1783; from 10 till 2½.*

The weather variable. Sometimes flying and thunder showers.

Barometer 30.0; Thermometer 64° at 3 P. M.

N ^o	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		wt		diam	gun	pend			<i>p</i>	<i>g</i>	<i>n</i>	
	oz	oz	dr	inches	inches	inches	inches	inch	lb	inches		feet
1	2				2.55							
2	2				2.50							
3	2				2.50							
4	16				24.6							
5	16				21.8							
6	16				24.5							
7	16	16	12½	1.96	36.0	19.6	88.3	8	663.0	77.35	40.20	1411
8	16	16	12½	1.96	36.7	19.8	88.6	10	664.6	77.38	40.19	1424
9	2				2.5							
10	16				28.25							
11	16				26.4							
12	16				24.7							
13	16	16	12½	1.96	39.1	23.2	87.8	11	666.3	77.41	40.19	1689
14	16	16	12½	1.96	35.8	21.7	88.5	10	667.9	77.44	40.19	1572
15	16	16	12½	1.96	37.9	23.3	91.1	11	669.6	77.47	40.19	1644
16	16	16	12½	1.96	40.7	24.8	90.6	11	671.2	77.50	40.19	1765
17	16	16	12½	1.96	42.4	24.2	91.3	10	672.9	77.53	40.18	1714

The GUN was n^o 1 in the first 8 rounds; and n^o 2 in the rest to the end. The weight, &c. as before.

The PENDULUM was a new block, made of found dry elm, painted, and hung in the same frame as the former; but turned end-ways, or the ends of the fibres towards the gun; whereas the former was side-ways.

It

It was firmly bound round with strong iron bars; but neither plates of iron nor lead were put within it. The dimensions of the block are,

Length from front to back	-	26 $\frac{3}{4}$ inches
Depth of the face	- - -	24 $\frac{3}{4}$
Breadth of the same	- - -	18 $\frac{1}{4}$
Its weight with iron	- - -	664 lb
Radius to tape as before	- -	117.8 inches
To center of gravity	- - -	77.35
Oscillations per minute	- -	40.20

At the 7th and 15th rounds the balls struck both in firm and solid wood, when their penetrations, to the hinder part of the ball, measured 10 $\frac{1}{2}$ and 11 inches; so that the fore part penetrated 12 $\frac{1}{2}$ inches in the first case, and 13 inches in the latter.

	Gun 1	Gun 2
Mean length of the charge	11.4	11.3
Mean recoil with ball - -	36.35	40.03 omitting n° 14
Ditto without - - - -	23.63	26.45
Difference, or $c =$ - - -	12.72	13.58
Hence velocity by the recoil	1399	1497
Mean ditto by the pendulum	1419	1676
Difference, the recoil less -	20	179
Or nearly the part - - -	$\frac{1}{71}$	$\frac{1}{9}$

58. N. B. In this day's experiments, and those that follow, as long as the same block of wood is used, the theorems for correcting the place of the center of gravity, and the number of oscillations per minute, as laid down at Art. 44, will be a little altered, when the weight of the pendulum is varied at the center of the block. The reason of which is, that now the distance to the center is 88.7, which before was only 88.3.

And by using 88.7 for 88.3 in the theorems in that article, those theorems will become

$$G = 88.7 - \frac{7524}{p} \text{ for the new value of } g, \text{ and}$$

$$N = 39.646 + \frac{314}{p-93} \text{ for the new value of } n.$$

Had i been = 89.3, the new value of g and n would have been

$$G = 89.3 - \frac{7920}{p}, \text{ and}$$

$$N = 39.51 + \frac{386}{p-100}.$$

And these last are the proper theorems for this day's experiments, the mean distance of the points struck being nearly 89.3.

59. *Wednesday, August 13, 1783; from 10 till 2.*

The weather cloudy and misty, but it did not rain.

Barometer 30.17; Thermometer 64° at 5 P. M.

N ^o	Powder	Ball's		Vibration of		Point struck	Plugs	Values of			Veloc. ball	
		wt		gun	pend			<i>p</i>	<i>g</i>	<i>n</i>		
		oz	dr									inches
1	2				2.5							
2	2				2.6							
3	16				27.6							
4	16				27.9							
5	16	16	12½	1.96	41.8	26.8	87.0	12	672.9	77.53	40.18	1992
6	16	16	12½	1.96	36.3	23.1	86.2	13	674.5	77.55	40.18	D
7	16	16	12½	1.96	42.3			11				
8	16	16	12½	1.96	41.0	25.2	84.7	11	677.8	77.58	40.18	1940
9	8				14.2							
10	8				13.5							
11	8				13.6							
12	8	16	12½	1.96	27.6	22.4	84.4	10	679.4	77.60	40.18	1735
13	8	16	12½	1.96	28.8	25.8	90.3		681.0	77.63	40.18	1872

The GUN was n° 3. In the 5, 6, 7, 8, and 12th rounds, the gun had from 15' to 20' elevation. At the 6th round an uncommon large quantity of powder came out unfired, so as to scatter a great way over the ground, and bespatter the face of the screen and pendulum very much; which was not the case in any other round. And this may account for the smaller arcs described at that number.

The PENDULUM was in the same condition as it had been left hanging after the last day's experiments, with all the balls and plugs in it. After this day's experiments, its weight was found to be 681 lb, including all the balls and plugs, except one which flew out behind the pendulum at the 7th round, occasioned by this ball striking in the same hole as n° 6, and knocking it out. This ball, which came out, was quite whole and perfect; it was black on the hinder part with the powder, but rubbed bright before with the friction in passing through the wood. The tape of the pendulum also broke at this round, so that the vibration could not be measured.

The value of i , or the mean among the distances of the point struck this day and the last is 88.

Of the plugs, this day and the last, 10 inches weighed 9 oz.

	8 oz	16 oz
Mean length of the charge -	6.0	11.1
Mean recoil with ball - -	28.2	41.7
Ditto without - - - - -	13.77	27.75
Difference, or $e =$ - - -	14.43	13.95
Hence velocity by the recoil	1594	1542
Mean ditto by the pendulum	1803	1966
Difference, the recoil less -	209	424
Or nearly the part - - -	$\frac{1}{9}$	$\frac{1}{3}$

60. Monday, September 8, 1783; from 10 till 1½ P. M.

Weather windy and cloudy, with some drops of rain.

Barometer 30.03; Thermometer 61° at 10 A. M.

N ^o	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		wt		diam	gun	pend			<i>p</i>	<i>g</i>	<i>n</i>	
	oz	oz	dr	inches	inches	inches	inches	inch	lb	inches		feet
1	2				2.7							
2	2				2.55							
3	2				2.6							
4	4				6.55							
5	4				6.1							
6	4				6.8							
7	4	16	13	1.96	17.4	17.8	88.1	10	663.0	77.35	40.20	1281
8	4	16	13	1.96	18.3	19.0	88.3	9	664.7	77.37	40.19	1369
9	4	16	13	1.96	18.2	18.8	88.0	8	666.3	77.40	40.19	1363
10	4	16	13	1.96	17.9	18.0	87.2	9	667.8	77.42	40.19	1321
11	2	16	13	1.96	10.9	12.4	87.8	8	669.4	77.44	40.19	906
12	2	16	13	1.96	11.2	12.5	85.8	7	670.9	77.47	40.19	937
13	2	16	13	1.96	11.0	12.5	86.1	7	672.4	77.49	40.19	936

The GUN n^o 3, with every circumstance as usual; except that in the last four rounds it had 15' elevation.

The PENDULUM had been repaired, the balls and plugs taken out, a square hole cut quite through, and a found piece fitted in; and the face covered with sheet lead as before.

Its weight at the beginning 663 lb

To the center of gravity - 77.35 inches

To the tape - - - - 117.8

The

The vibration at n° 8 a little doubtful, as the tape broke.

The plugs weighed 1 oz per inch.

The value of i , or the mean distance of the points struck, 87.3.

Weight of Powder	- - - - 2 oz.	4 oz.
Mean length of the charge	- 1.9	3.2
Mean recoil with ball	- - 11.03	17.95
Ditto without	- - - - 2.62	6.48
Difference, or $c =$	- - - 8.41	11.47
Hence velocity by the recoil	- 928	1266
Mean ditto by the pendulum	926	1334
Difference, recoil less,	- - - 2	- 68
Or nearly the part	- - - $\frac{1}{463}$	$\frac{1}{18}$

61. Wednesday,

61. *Wednesday, September 10, 1783; from 10 till 12.*

The weather was fine.

Barometer 29·7; Thermometer 60° at 10 A. M.

N ^o	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		wt		diam	gun	pend			p	g	n	
		oz	dr	inches			inches	inches				inch
1	2				2·5							
2	2				2·2							
3	2				2·45							
4	4				5·8							
5	4				5·8							
6	4				5·7							
7	8				12·1							
8	8				12·1							
9	8				12·2							
10	8	16	12½	1·96	24·5	18·0	88·3	5	671·4	77·48	40·19	1315
11	8	16	12½	1·96	25·1	19·3	89·5	8	672·8	77·50	40·19	1394
12	8	16	12½	1·96	24·8	18·1	86·8	6	674·3	77·52	40·19	1351
13	4	16	12½	1·96	15·85	14·65	88·5	7	675·7	77·54	40·19	1075
14	4	16	12½	1·96	15·7	14·1	87·4	6	677·2	77·56	40·19	1050
15	4	16	12½	1·96	16·35	15·4	88·5	3	678·6	77·59	40·18	1136
16	2	16	12½	1·96	10·0	10·75	89·3	4	679·8	77·61	40·18	787
17	2	16	12½	1·96	9·9	10·6	89·8	3	681·1	77·63	40·18	774
18	2	16	12½	1·96	10·1	10·65	88·0	3	682·3	77·65	40·18	795

The GUN, n^o 1. Weight and other circumstances as usual.

The PENDULUM as left hanging since Monday. Its radius, &c. as usual.—The value of *i*, or the mean distance among the points struck this day and the former, is 88·0.

The plugs weighed 1 oz per inch.

Weight

Weight of Powder	- - 2 oz	4 oz	8 oz
Mean length of the charge	- 1.9	3.2	5.7
Mean recoil with ball	- - 10.00	15.97	24.8
Ditto without	- - - - 2.38	5.77	12.1
Difference, or $c =$	- - - 7.62	10.20	12.7
Hence velocity by the recoil	838	1122	1396
Mean ditto by the pendulum	785	1087	1353
Difference, the recoil more,	- 53	35	43
Or nearly the part	- - - $\frac{1}{15}$	$\frac{1}{31}$	$\frac{1}{31}$

62. Thursday, September 11, 1783; from 10 till 12.

The weather was fine.

Barometer 29.93; Thermometer 60° at 10 A. M.

N ^o	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		wt		diam	gun	pend			p	g	n	
	oz	oz	dr	inches	inches	inches	inches	inch	lb	inches		feet
1	2				2.65							
2	2				2.7							
3	2				2.65							
4	4				6.2							
5	4				6.1							
6	4				6.0							
7	8				13.7							
8	8				13.1							
9	8				14.1							
10	8	16	12 $\frac{1}{2}$	1.96	27.0	21.2	87.4	4	681.2	77.63	40.18	1590
11	8	16	12 $\frac{1}{2}$	1.96	27.1	21.3	88.1	6	682.5	77.65	40.18	1589
12	8	16	12 $\frac{1}{2}$	1.96	26.3	20.2	86.7	12	683.9	77.67	40.18	1535
13	4	16	12 $\frac{1}{2}$	1.96	17.3	16.6	87.5	9	685.7	77.70	40.18	1253
14	4	16	12 $\frac{1}{2}$	1.96	17.1	16.7	89.9	8	687.3	77.72	40.18	1230
15	4	16	12 $\frac{1}{2}$	1.96	17.1	16.7	89.9	7	688.8	77.75	40.17	1233
16	2	16	12 $\frac{1}{2}$	1.96	10.3	11.5	90.1	4	690.4	77.77	40.17	849
17	2	16	12 $\frac{1}{2}$	1.96	10.45	11.7	90.3	3	691.7	77.80	40.17	864
18	2	16	12 $\frac{1}{2}$	1.96	10.3	11.5	89.9	2	692.9	77.82	40.17	855

The

The GUN n° 2. In the last 5 rounds it had about 10' depression.

The PENDULUM the same as left hanging since yesterday. After the experiments were concluded to-day, it weighed 694lb.—The plugs weighed 1 oz per inch.

The weight of balls and plugs lodged in the block, these last three days, was 36lb; which added to 663; the weight at the beginning, makes 699: but it weighed at the end only 694; so that it lost 5 lb of its weight in the 4 days, or $1\frac{1}{4}$ lb per day on a medium.

The value of i , or the mean among the distance of the points struck these three days, is 88.3.

	2 oz	4 oz	8 oz
Mean length of the charge -	1.8	3.1	5.7
Mean recoil with ball - -	10.35	17.17	26.80
Ditto without - - - -	2.67	6.10	13.63
Difference or $c =$ - -	7.68	11.07	13.17
Hence velocity by the recoil	846	1220	1452
Mean ditto by the pendulum	856	1239	1571
Difference, the gun less, -	10	19	119
Or nearly the part - - -	$\frac{1}{86}$	$\frac{1}{65}$	$\frac{1}{13}$

63. Tuesday,

63. Tuesday, September 16, 1783; from 12 till 2.

The weather was rainy.

Barometer 29.9; Thermometer 64° at noon.

N ^o	Powder	Vibration of							
		gun							
	oz								
1	2			2.3					
2	2			2.5					
3	2			2.35					
4	4			5.25					
5	4			5.05					
6	4			5.4					
7	8			11.65					
8	8			11.9					
9	8			12.05					
10	12			17.3					
11	12			19.3					
12	12			18.7					
13	12			17.1					
14	16			25.3					
15	16			23.3					
16	16			24.0					
17	20			28.5					
18	20			28.2					
19	20			24.8					

The GUN was n^o 1.

The last n^o very uncertain; the tape, being very wet, twisted, and was entangled.

	2 OZ	4 OZ	8 OZ	12 OZ	16 OZ	20 OZ
Mean length of charge - -	1.9	3.2	5.6	8.2	10.6	13.2
Mean recoil, omitting n ^o 19,	2.38	5.23	11.9	18.1	24.2	28.2

64. Thursday, September 18, 1783; from 10 till 3 P. M.

The weather fair and mild.

Barometer 30.08; Thermometer 64° at 10 A. M.

N ^o	Powder		Ball's		Vibration of		Point struck	Plugs	Values of			Velocity of the ball
	wt	ht	wt	dr	gun	pend			p	g	n	
	oz	inches			inches	inches			lbs	inches		
1												
2												
3												
4												
5												
6												
7	24	14.5	16	12½	38.6	17.3	90.8	14	655.0	77.21	40.21	1194
8	32	21.6	16	12½	44.0	13.0	92.9	8	656.8	77.24	40.20	880
9	36	24.4	16	12½	45.8	12.3	92.5	7	658.3	77.27	40.20	838
10	39	27.2	16	12½	47.5		went over		659.7	77.29	40.20	
11	20	13.3	16	12½	36.7	15.5	85.8	11	660.8	77.29	40.20	1144
12	12	8.1	16	12½	29.9	18.75	86.8	11	662.5	77.32	40.20	1371
13	14	9.3	16	12½	27.3	16.2	85.8	11	664.2	77.35	40.20	1202 D
14	10	6.9	16	13	28.5	19.15	86.6	10	665.9	77.37	40.20	1409
15	14	10.1	16	13	31.2	19.0	89.5	10	667.5	77.40	40.19	1357
16	16	11.1	16	13	32.7	18.0	91.4	5	669.1	77.43	40.19	1262 D
17	8	5.7	16	13	26.4	20.05	89.7	9	670.4	77.45	40.19	1436
18	6	4.6	16	13	20.7	17.0	88.5	9	671.9	77.48	40.19	1237
19	12	8.4	16	13	29.0	18.55	89.9	7	673.4	77.51	40.19	1333
20	10	6.9	16	13	28.8	20.05	90.5	6	674.8	77.53	40.19	1434
21	14	9.6	16	13	32.2	18.5	91.1	6	676.2	77.56	40.19	1318
22	8	5.5	16	13	25.3	18.8	89.9	6	677.6	77.59	40.18	1360
23	12	8.4	16	13	31.1	19.8	90.9	6	678.9	77.61	40.18	1420
24	10	7.0	16	13	28.6	19.35	89.7	6	680.2	77.64	40.18	1409
25	8	5.7	16	13	25.4	18.5	89.5	5	681.5	77.67	40.18	1354
26	16	10.7	16	13	31.2	16.75	89.3	5	682.8	77.69	40.18	1231 D
27	16	11.1	16	13	31.5	17.2	91.4	5	684.1	77.72	40.18	1238 D
28	14	9.7	16	13	33.4	20.0	90.5	4	685.4	77.75	40.17	1457

The GUN n° 1. The charge of powder was gradually increased till the gun became quite full at n° 10, when there was just room for half the ball to lie within the muzzle; which being too short a length to give a direction to the ball, it missed the pendulum, going over and just striking the top of the screen frame, about $21\frac{1}{2}$ inches above the line of direction, which, though a very slender piece of wood, turned the ball up into a still higher direction, in which it struck the bank over the pendulum, and entered it sloping, though but a little way: all which circumstances shew that the force of the ball was but small. And even at the 9th round, when the center of the ball was about 3 inches within the gun, the ball struck the pendulum 5 inches out of the line of direction. The gun was scarce ever sensibly heated.

The diameter of the balls 1.96 inches.

The PENDULUM had been gutted, and had received a new core. It was hung up in the morning of the day before yesterday, when it weighed 659 lb. And when taken down this evening it weighed only 686 lb, which is near 4 lb less than the balls and plugs ought to make it; and which 4 pounds must have evaporated in the 3 days.

The plugs weighed $\frac{7}{8}$ of an ounce to the inch.

The value of i , or mean point struck, 89.7.

All the three rounds with 16 oz are very doubtful, and seem to be too low, from some unknown cause.

Mean velocity by the pendulum, &c.

Powder		Recoil		Veloc.	
wt	ht	gun		ball	
8	- 5.6	- -	25.7	- -	1383
10	- 6.9	- -	28.6	- -	1417
12	- 8.3	- -	30.0	- -	1375
14	- 9.7	- -	32.3	- -	1333
16	- 11.0	- -	31.8 D	- -	1243 D
20	- 13.3	- -	36.7		
24	- 14.5	- -	38.6		
32	- 21.6	- -	44.0		
36	- 24.4	- -	45.8		
39	- 27.2	- -	47.5		

A a 2

65. Thursday,

65. *Thursday, September 25, 1783; from 10 A. M. till 3 P. M.*

Fine, clear, and warm weather.

Barometer 29.93; Thermometer 59° at 10 A. M.

N ^o	Powder		Ball's		Vibration of		Point	Plugs	Values of			Veloc.
	wt	ht	wt	dr	gun	pend	struck		<i>p</i>	<i>g</i>	<i>n</i>	ball
	oz	inches	oz	dr	inches	inches	inches	inch	lb	inches		feet
1	8	5.9	16	13	27.6	23.8	89.3	14	643.0	77.00	40.22	1632
2	10	7.2	16	13	29.5	23.0	88.7	15	644.9	77.03	40.22	1593
3	12	8.4	16	12½	32.0	22.0	88.7	15	646.8	77.06	40.21	1532
4	14	9.4	16	12½	32.2	21.3	88.5	15	648.8	77.09	40.21	1491 D
5	16	11.3	16	12½	39.4	23.6	88.3	13	650.8	77.12	40.21	1662
6	18	12.3	16	12½	37.1	21.0	89.0	12	652.6	77.16	40.21	1472
7	20	13.2	16	12	39.8	21.9	91.6	17	654.4	77.19	40.21	1499
8	22	15.1	16	12	41.5	21.7	91.5	11	656.5	77.22	40.20	1492 D
9	24	15.8	16	12	42.6	21.3	91.6	10	658.2	77.25	40.20	1468
10	28	18.9	16	12	44.2	18.1	90.5	10	659.8	77.28	40.20	1266 D
11	32	22.1	16	12	52.8	20.3	90.8	9	661.5	77.31	40.20	1419
12	8	5.5	16	12	27.0	22.2	90.5	8	663.1	77.35	40.20	1562
13	10	7.0	16	12	30.5	22.9	90.0	7	664.6	77.38	40.20	1624
14	12	8.1	16	12	32.4	21.8	85.2	15	666.1	77.41	40.19	1638
15	14	9.3	16	12	32.9	20.4	86.4	15	668.0	77.44	40.19	1517
16	16	10.9	16	12	39.0	22.2	85.8	13	670.0	77.47	40.19	1667
17	8	5.5	16	12	25.2	19.6	87.9	15	671.8	77.50	40.19	1441 D
18	8	5.5	16	12	26.3	20.8	89.2	13	673.6	77.53	40.19	1512
19	10	6.7	16	12	26.3	19.6	89.1	13	675.4	77.56	40.18	1431 D
20	12	8.2	16	12	32.7	22.8	88.8	11	677.2	77.59	40.18	1675
21	14	9.1	16	12	35.1	17.7	89.0	11	678.9	77.63	40.18	1302 D
22	16	10.4	16	12	32.8	15.6	89.1	8	680.6	77.66	40.18	1149 D
23	6	4.4	16	12	22.8	14.5	89.1		682.1	77.69	40.18	1070 D

The GUN was n^o 2.

The diameter of the balls 1.96 inches.

The

The PENDULUM had been repaired with a new core, but of very soft and damp wood. It was hung up yesterday morning, when it weighed 653 lb. And when taken down this evening it weighed only 678 lb with all the balls and plugs, the whole ball which came out behind, as well as the broken pieces of the wood and balls which flew out in the latter rounds, being collected and weighed with it; which is about $15\frac{1}{2}$ lb less than it ought to be; so that about $15\frac{1}{2}$ lb has been lost by evaporation in the space of 30 hours, or about half a pound an hour.

At n^o 4, 8, 10 the tape of the pendulum entangled and broke, which rendered those vibrations doubtful, as marked D. Some other rounds are marked doubtful, from some other cause, perhaps the badness of the wood in the pendulum, which split very much; from which circumstance part of the force of the ball might be lost by the lateral pressure.

The plugs weighed 14 oz to 15 inches.

The value of i , or the mean point struck, 89.5 inches.

The penetration at the 1st and 7th rounds, which were made in fresh parts of the wood, were from 19 to 20 inches; so that the fore part of the ball penetrated about $21\frac{1}{2}$ inches in this soft wood.

Mean recoil and velocity by the pendulum.

Powder			Recoil			Veloc.
8	-	-	26.7	-	-	1569
10	-	-	30.0	-	-	1608
12	-	-	32.4	-	-	1615
14	-	-	33.4	D	-	1517 D
16	-	-	36.8	D	-	1664 D
18	-	-	37.1			
20	-	-	39.8			
22	-	-	41.5			
24	-	-	42.6			
28	-	-	44.2			
32	-	-	52.8	D		

But

But these mediums are not much to be depended on, as the velocities are all very irregular. It is, in particular, highly probable that the velocity here found for 14 oz of powder is too small, and that for 16 oz too great.



66. *Monday, September, 29, 1783; from 10 A. M. till 1½ P. M.*

The weather fine, clear, and warm.

Barometer 30.28; Thermometer 64° at 10 A. M.

N ^o	Powder		Ball's wt		Vibration of		Point struck	Plugs	Values of			Veloc. ball
	wt	ht			gun	pend			<i>p</i>	<i>g</i>	<i>n</i>	
	oz	inches	oz	dr	inches	inches	inches	inch	lb	inches		feet
1	2				2.8							
2	2				2.75							
3	6	4.5	16	11½	22.6	20.5	88.9	7	654.0	77.20	40.21	1448
4	8	5.6	16	11½	26.9	22.1	89.1	10	655.4	77.22	40.21	1561
5	10	6.9	16	11½	30.2	23.4	91.3	8	656.9	77.25	40.20	1618
6	12	8.3	16	11½	33.3	23.9	90.6	9	658.3	77.27	40.20	1669
7	14	9.5	16	11½	37.4	24.7	88.9	10	659.8	77.30	40.20	1763
8	16	10.7	16	11½	35.9	21.5	87.3	9	661.3	77.32	40.20	1566 D
9	16	11.0	16	11½	40.1	23.5	87.8	8	662.8	77.35	40.20	1707
10	18	12.1	16	11½	32.7	18.3	87.1	6	664.2	77.37	40.20	1343 D
11	18	12.2	16	11½	39.5	21.9	87.7	12	665.5	77.40	40.20	1598
12	20	13.0	16	11	42.7	23.4	91.8	10	667.1	77.43	40.19	1639
13	14	9.6	16	11		21.6	89.3	9	668.6	77.46	40.19	1561

The GUN n^o 2.—At the last round the tape broke, so the recoil could not be measured. N^{os} 8 and 10 are plainly both irregular, the recoils being greatly deficient: the vibrations of the pendulum might perhaps be defective by the balls being resisted sideways by the wood, or by changing

changing their direction within the block; but there is no cause which I can suspect for the defective recoils of the gun, as all the circumstances were alike in every case, and the heights of the charges shew that there was no mistake in the quantity of powder.—At the last firing the vent had a small channel blown in it, though the gun was no where very hot.

The PENDULUM had received a new core of sound dry elm, and weighed this morning, when it was hung up, 654lb.

The diameter of the balls 1.96 inches.

The plugs weighed $6\frac{1}{4}$ oz to 8 inches.

The value of i , or mean point struck, 89.1.

The first penetration was 12 inches, measured behind the ball, and consequently the fore part penetrated 14 inches.

Mean recoil of gun and velocity of ball :

Powder			Recoil			Veloc.
6	-	-	22.6	-	-	1448
8	-	-	26.9	-	-	1561
10	-	-	30.2	-	-	1618
12	-	-	33.3	-	-	1669
14	-	-	37.4	-	-	1662
16	-	-	38.0	-	-	1637
18	-	-	39.5	-	-	1598
20	-	-	42.7	-	-	1639

67. Thursday,

67. *Thursday, September 30, 1783; from 10 A. M. till 1½ P. M.*

Fine, clear, and warm weather.

Barometer 30.25; Thermometer 64° at 10 A. M.

N ^o	Powder		Balls		Vibration of		Point struck	Plugs	Values of			Veloc. of the ball
	wt oz	ht inches	wt oz	dr	gun inches	pend. inches			p lb	g inches	n	
1	2	1.9			2.4							
2	2	1.9			2.6							
3	10	7.1	16	14	27.5	19.4	89.6	6	669.0	77.46	40.19	1383
4	12	8.4	16	14	31.9	20.4	88.7	6	670.3	77.48	40.19	1472
5	8	5.8	16	14	25.3	18.7	88.8	6	671.6	77.51	40.19	1351
6	14	9.3	16	14	32.7	19.4	88.9	7	672.9	77.53	40.19	1403
7	6	4.6	16	14	21.6	18.5	90.3	6	674.3	77.55	40.19	1320
8	10	6.9	16	14	27.5	19.8	91.2	7	675.6	77.58	40.19	1403
9	12	8.3	16	14	29.5	19.9	92.2	5	677.0	77.60	40.18	1398
10	8	5.7	16	14	25.3	18.9	90.2	6	678.3	77.62	40.18	1360
11	14	9.6	16	14	32.6	19.7	91.2	6	679.6	77.65	40.18	1405
12	6	4.4	16	14	21.9	18.0	87.0	7	680.9	77.67	40.18	1349
13	10	6.9	16	14	28.7	18.8	86.5	7	682.3	77.69	40.18	1420
14	12	8.4	16	14	31.7	20.0	87.9	6	683.7	77.71	40.18	1490
15	8	6.0	16	14	26.4	19.4	88.0	7	685.0	77.74	40.17	1447
16	14	9.3	16	14	32.2	18.4	86.5	7	686.4	77.76	40.17	1399
17	6	4.3	16	14	21.7	17.9	89.2	5	687.8	77.78	40.17	1323

The GUN n^o 1.—The vent blew a little, though the gun was never very warm.

The PENDULUM was the same as it hung since yesterday, with all the balls in it; but the other end of it was turned, which bore the fi ings very well, the core being of sound dry wood. At the end of the experiments this day the pendulum weighed 689 lb, which is only 1 lb less

less than it ought to be by the addition of the balls and plugs to the first weight; so little was it less of weight by evaporation, owing to the dryness of the wood.

The diameter of the balls 1.96 inches.

The plugs weighed $6\frac{1}{2}$ oz to 8 inches.

The value of i , or mean point struck, 89.1 inches.

The first penetration, being in sound wood, was $14\frac{1}{4}$ inches to the fore part of the ball.

This set of experiments, as well as those of the three preceding days, were made to determine the best charge, or that which gives the greatest velocity.

This is a good set of experiments, and the

Mean recoil, and velocity of the ball by the pendulum, are as follows :

Powder	Recoil	Veloc.
6 - -	21.7 - -	1331
8 - -	25.6 - -	1386
10 - -	27.9 - -	1402
12 - -	31.0 - -	1453
14 - -	32.5 - -	1402

which velocities, as well as the recoils, are found by adding those of each sort together, and dividing by the number of them, as below :

6	8	10	12	14
1320	1351	1383	1472	1403
1349	1360	1403	1398	1405
1323	1447	1420	1490	1399
<u>3) 3992</u>	<u>4158</u>	<u>4206</u>	<u>4360</u>	<u>4207</u>
means 1331	1386	1402	1453	1402

where the velocity with 12 oz is greatest.

The end of Experiments in 1783.

THE EXPERIMENTS OF 1784.

68. *Wednesday, July 21, &c. 1784.*

IN the course of last year's operations we experienced several inconveniences from some parts of our apparatus, which we determined to remedy if possible. These regarded chiefly the time-pieces, the axes of vibration, and the method of measuring by the tape. For measuring the time of a certain number of vibrations, we united the use of a second stop watch with a simple half-second pendulum, made of a leaden bullet suspended by a filken thread, which did not always agree together. Again, the axes of the gun and pendulum frames were not found to be so devoid of friction as might be wished. But, above all, the chief cause of dissatisfaction, was the method of measuring the extent of the vibrations by means of the tape; which was, notwithstanding all possible care and precaution, still subject to much irregularity, by being wetted by rain, or blown aside by the wind, or otherwise entangled, which rendered the measurements doubtful and irregular.

The preceding part of this year therefore was employed in correcting these and other smaller imperfections in the apparatus. To our time-pieces we added a peculiar one, which measures time to 40th parts of a second.—Next, by a happy contrivance, the friction of the axes was almost intirely taken off. This was effected by means of sockets of a peculiar construction, for the axes to work in. First imagine the half of a short cylinder, of 2 or 3 inches long, cut lengthways through the axis, and of a diameter a very little more than the ends of the axis that are intended to work in it: if this were all, it is evident that the axis, in
vibrating,

vibrating, would touch this socket in one line only, because their diameters were unequal. Next imagine the inside of this socket to be gradually ground down towards each end, from nothing in the middle; so that the inside resembled a tube having its two ends bent downwards, and rising highest in the middle. Then it is evident that the axis will touch the socket in this one middle point only. And farther, the under sides of the axis itself were ground a little, to bring the undermost line to an edge, something like the pivots of a scale beam. The consequence was, that the friction was not sensible in a great number of vibrations; and hereafter we commonly made the gun and pendulum vibrate for just 10 minutes, and divided the counted number of vibrations by 10, for the mean number per minute—And for measuring the arcs of vibration more certainly and accurately, we have constructed a strong wooden circular arch, of about 4 feet in length, cut out to a radius of just 10 feet. This arch is divided into chords of equal parts, each the 1000th part of the radius, or $\frac{12}{1000}$ th parts of an inch, as before described in Art. 16. This arch being placed 10 feet below, and concentric with the axis, and the groove in the middle of it filled with the soft composition of soap and wax, the stylette, or small sharp spear, traces in the groove the extent of the vibration, and the corresponding divisions on each side of the groove shew the length of the chord vibrated. And as these chords are in 1000th parts of the radius, the value of r , in the theorem for the velocity of the ball, will be 1000 for all the following experiments; and then that theorem will become

$$v = \frac{59}{96} \times \frac{p+b}{bin} gc \text{ by the pendulum, and}$$

$$v = \frac{59}{96} \times \frac{g g c}{bin} \text{ by the recoil of the gun.}$$

Or $v = 12.742 \times \frac{c}{b}$ or $\frac{51}{4} \times \frac{c}{b}$ by the gun n° 2, when we substitute the values of g , g , i , n , specified in Art. 36. And farther, when $b = 1.047$, it is $v = 12\frac{1}{6}c$.

The apparatus having been prepared, we employed the three days, July 21, July 26, and August 3, in hanging it up, and in weighing, measuring, and adjusting all the parts, and trying them by firing a few rounds with powder only. The 4 rounds fired on the first of those days, of 4 ounces each, with the gun n° 1, weighing 917lb, gave 56 at the first round, and at each of the other three 57 divisions on the measuring arc, for the recoil of the gun.

69. *Wednesday, August 4, 1784.*

Frequent showers of rain.

N ^o	Powder	Weight of			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		ball		gun	gun	pend			p	g	n	
	oz	oz	dr	lb			inches	inch	lb	inches		feet
1	2			478	55							
2	2			478	55							
3	2			478	57							
4	2			478	57							
5	4			478	122							
6	8			478	252							
7	6	16	14	478	426	166	88.9	9	631.5	76.79	40.23	1313
8	6	16	14	478	387	151	89.3	5	632.8	76.81	40.23	1192 D
9	6	16	14	478	426	164	88.9	8	634.0	76.83	40.23	1303
10	6	16	14	650	279	158	89.2	7	635.3	76.85	40.23	1254
11	6	16	14	650	290	160	88.0	8	636.6	76.87	40.23	1291
12	6	16	14	917	193	157	87.3	9	637.9	76.90	40.22	1280
13	6	16	14	917	199	162	86.9	9	639.1	76.92	40.22	1329
14	6	0	0	917	85	0						
15	6	1	10	917	104	20	82.3		640.1	76.94	40.22	
Omitting n° 8, the mean is												1295

Here, and in all the future days, the chords of vibration, of both gun and pendulum, are expressed in 1000th parts of the radius.

The GUN was n° 1.—We began this day with the weight of the gun and its iron frame only, without any of the leaden weights. Then the one set of weights was put on at n° 10, and the other at n° 12. This was done to try the effects of different weights of gun on the velocity of the ball, experimentally to correct a common error which had been adopted from time immemorial, by professional men, namely, that heavier guns, *cæteris paribus*, give the greater velocities. The erroneoufness of which opinion is proved by the experiments of this and some of the following days. And it is needless to prove *a priori* to scientific men, that the difference in the effects cannot be rendered sensible by any measurements which we can make of the velocity.

The PENDULUM was the block of last year, with a new core, and a facing of sheet lead. Its weight, taken this morning, was 627 lb.

The plugs weighed 7 ounces to 11 inches, on an average; which proportion may always be used in future, at least till another be mentioned.

The 8th n° is doubtful, and is omitted in the medium.

The 14th was with powder only, like the first six. And the 15th was without ball, having only a wad made of junk, weighing 1 oz 10 dr. This made a small impresson, of about half an inch deep, in the face of the pendulum, and rebounded back. And it struck the pendulum at more than 6 inches above the line of direction.

Note, the center of the pendulum, as before, is at 88.7 inches below the axis. And the value of *i*, for the mean distance of the points struck, is 88.4.

By comparing together the first 6 rounds, which are all with the same weight of gun, we find that the mean proportion of the recoil, with the different charges, without balls, is as follows :

2 oz	4 oz	8 oz
56	122	252

the recoils being rather in a higher proportion than the charge of powder.

If

If we compare the mean of the first 4, with 2 oz of powder and 478 lb weight of gun, with the mean of July 21, with 4 oz of powder and 917 lb weight of gun, we shall obtain as follows :

Charge	- -	2 oz		4 oz
Weight of gun	478 lb	- -	917 lb	
Recoil	- -	56	- -	57

So that, in this instance, the less charge gives a recoil in proportion to the greater charge, a little above the direct ratio of the weight of powder, and inverse ratio of the weight of the gun. For that ratio, or 2×917 to 4×478 , is as 56 to 58.

If we compare n° 5 with the mean of July 21, which are both with 4 oz of powder, they will stand thus :

Weight of gun	478	- -	915
Recoil	- -	122	- - 57

which shews that, in this instance, the same charge gives more than double the recoil to half the weight of the gun.

Lastly, if we compare the means of each pair of velocities with the several weights of gun, we shall have as follows :

1313	}	1308	mean	with	478 lb	wt of	gun
1303							
1254	}	1273	- - -	650	- - -		
1291							
1280	}	1305	- - -	917	- - -		
1329							

which differences are neither regular, nor greater than happen from different trials with the weight and all other circumstances the same.

For the 6 oz charge the

Mean recoil with ball	-	196
Ditto without	- - -	85
Difference, or c	= -	111
Hence velocity by recoil	1339	
Ditto by the pendulum	-	1295
Difference	- - - -	44
Or the part	- - - -	$\frac{1}{36}$

70. *Thursday, August 5, 1784.*

A fine warm day.

Barometer 29.98; Thermometer 68 at 10 A. M.

N ^o	Powder	Weight of			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		ball		gun	gun	pend			<i>p</i>	<i>g</i>	<i>n</i>	
	oz	oz	dr	lb			inches	inch	lb	inches		feet
1	4			485	127							
2	6			485	176							
3	6	16	14	485	460	195	86.8	9	640.4	76.94	40.22	1606
4	6	16	14	485	459	197	87.2	8	641.7	76.96	40.22	1619
5	6	16	14	655	312	196	88.1	7	642.9	76.99	40.22	1598
6	6	16	14	655	319	196	88.3	7	644.2	77.01	40.22	1598
7	6	16	14	917	218	200	87.3	9	645.5	77.03	40.21	1653
8	6	16	14	917	216	196	88.3	9	646.7	77.06	40.21	1605
9	6	16	14	1170	174	202	89.4	7	648.0	77.08	40.21	1637
10	6	16	14	1170	168	198	89.1	8	649.3	77.11	40.21	1614
mean of all												1616

The GUN was n^o 3.—Began first with its own weight only; then at n^o 5 put on one pair of the usual weights; at n^o 7 the other pair; and lastly at n^o 9 fixed on some extra weights. But the result shews that the velocity of the ball is the same with all of them.

The PENDULUM as left hanging since yesterday.

The value of *i*, or medium among the points struck these last two days, is 88.2.

1606 } 1613 mean velocity with 485 lb weight of gun
1619 }

1598 } 1598 - - - - - 655 - - - - -
1598 }

1653 } 1629 - - - - - 917 - - - - -
1605 }

1637 } 1625 - - - - - 1170 - - - - -
1614 }

1616 mean for 6 oz with gun 3.

71. *Saturday,*

71. *Saturday, August 7, 1784; from 11 till 2.*

The weather fair, but cloudy at times.

Barometer 29.92; Thermometer 64° at 2 P. M.

N ^o	Powder	Weight of				Vibration of		Point struck	Plugs	Values of			Veloc. ball
		ball		wad		gun	pend			<i>p</i>	<i>g</i>	<i>n</i>	
	oz	oz	dr	oz	dr			inches	inch	lb	inches		feet
1	4					58							
2	6			2	10	119	31	93.0					
3	6			2	9	120	20	78.0					
4	6					93							
5	6					93							
6	6			2	10	225	206	89.7	4	651.6	77.14	40.21	
7	6	16	14	2	8	228	212	89.4		652.9	77.16	40.21	
8	6	16	14	2	9	230	206	87.6	5	654.1	77.19	40.21	
9	6	16	14	2	9	232	217	89.8	12	655.4	77.21	40.21	
10	6	16	14	2	8	231	206	88.8	10	656.7	77.23	40.20	
11	6	16	14			219	199	89.2	6	657.9	77.26	40.20	
12	6	16	14	4	14	236	205	89.7	8	659.2	77.28	40.20	
13	6	16	14	4	12	235	220	89.5	5	660.5	77.30	04.20	

The GUN, n^o 3, with the usual leads, weighed 917 lb.

The mean height of the charge of 6 ounces was 4.5 inches.

The PENDULUM, as left hanging since the last day.

The value of *i*, or the mean among the points struck these last three days, 88.5.

The object of this day's business, was to try the effect of different degrees of ramming the charge of powder, with the effect of wads placed in

in different positions. Sometimes the powder was only set up without being compressed, and sometimes it was rammed with a different number of strokes, and pushed with various degrees of force: but no sensible difference was produced in the velocity. The wads, which were of 2 inches length, firmly made of junk or rope-yarn, and made large to be with difficulty pushed into the gun, were diversly placed and varied in number, being sometimes introduced between the powder and ball, and sometimes over both. But no effect was perceived from them on the velocity of the ball; this being indifferently the same, either with one wad, or two, or none at all. The reason of which is probably because the balls had very little windage. At the last two numbers two wads were used; in most of the others only one; weighing on an average about 2 oz 9 dr.

When balls were used with the wads, it was common for them both to enter the pendulum by the same hole. But it is remarkable that, when the wads were discharged without balls, they commonly struck wide of the line of the gun by 6 or 8 inches, and indifferently either too high or too low, or to the right or left; and sometimes they flew in pieces before they struck the block.

The velocities of the ball in these experiments are not computed, as the effects of the blow from the ball and the wad are compounded together, and that in an unknown degree, as the wad sometimes flies in pieces, and sometimes not, or strikes the pendulum with divers degrees of force at different times; and also sometimes the wads enter the pendulum, and sometimes they rebound from it.

72. *Tuesday, August 10, 1784; from 12 till 2.*

The weather thick and cloudy.

N ^o	Powder	Weight of				Vibration of		Point struck	Plugs	Values of			Veloc. ball
		ball		wad		gun	pend			<i>p</i>	<i>g</i>	<i>n</i>	
	oz	oz	dr	oz	dr			inches	inch	lb	inches		
1	4					56							
2	4					56							
3	6	14	2½			203	170	89.4	7	674.8	76.62	40.26	
4	6	14	3			195	159	89.8	8	676.1	76.64	40.26	
5	6	14	2½	4	12	214	161	89.5	5	677.3	76.67	40.26	
6	6	14	3½	4	10	215	163	89.3	6	678.6	76.69	40.26	
7	6	14	4	2	5	208	171	89.3	6	679.9	76.72	40.26	
8	6	14	2½	2	4	208	171	89.1	6	681.1	76.75	40.25	
9	6	13	15½	2	10	208	176	89.5	5	689.4	76.77	40.25	
10	6	14	3½	4	10	230	187	90.3	8	683.7	76.80	40.25	
11	6	14	3½	4	12	232	218	90.8	7	683.0	76.82	40.25	

The mean diameter of the ball was 1.875; so that the windage was 15.

The mean height of the charge of powder was 4.4.

The GUN n^o 3; its weight 917 lb.

The object this day was the effect of windage with low balls, and the effect of wads, both high and low ones. The wads struck variously, either above or below or with the ball. The two wads in the last round were made of well-twisted twine, and firmly bound: they struck the pendulum very hard blows. The other wads were of junk, and did not strike so hard.

Here,

Here, the balls being smaller, and consequently the windage more, the vibrations are much smaller, although wads were used. So that it seems the wads do not prevent the escape of the inflamed powder by the windage, nor make any sensible alteration in the velocity of the ball.

The velocities are not computed, for the same reason as specified in the last day's experiments.

73. The PENDULUM block had not been altered since the last day's experiments. But the iron stays of the stem had been changed for others that are stronger, and which weigh 10lb more than the old ones did. And this additional 10lb of iron must be added to the weight of the pendulum; and new theorems must be made out for determining the change in the center of gravity and the number of vibrations per minute. Now this rod, of uniform thickness, reached from the lower side of the axis to within 24 inches of the top of the block; consequently its length was 51.4 inches, and its middle point, or center of gravity, was at 26.6 inches below the middle of the axis of vibration. And this number 26.6 will be the value of i in the theorem $G = 77.3 + \frac{i - 77.3}{660 + b} b$ for the place of the new center of gravity, where the value of b is 10; which theorem gives

$$G = 77.3 - 0.76 = 76.54 \text{ for the center of gravity.}$$

And the same values of i and b , substituted in the theorem

$$N = 40.2 - \frac{40.2bi(i - 86.3)}{8805707 + bi(i + 86.3)}, \text{ give}$$

$$N = 40.2 + .07 = 40.27 \text{ for the number of oscillations.}$$

Hence then, in this new state of the pendulum, the value of g is 76.54, and the value of n 40.27, corresponding to the value 670 of p , or weight of the pendulum. That is

c c 2

p

p	g	n
670	76.54	40.27

are the new radical corresponding values of p , g , n . And these values, being substituted in the two general theorems, namely,

$$G = g + \frac{i-g}{p+b} b, \text{ and}$$

$$N = n - \frac{bin(inn - 140850)}{281700pg + bi(inn + 140850)},$$

they become

$$G = 76.54 + \frac{i-76.54}{670+b} b, \text{ and}$$

$$N = 40.27 - \frac{40.27bi(i-86)}{8820313 + bi(i+86)},$$

$$\text{or } N = 40.27 - \frac{i-86}{217947} bi \text{ nearly. Which are the theo-}$$

rems to be used now and hereafter for the values of g and n . And where the distance of the center of oscillation, answering to the number 40.47, is 86.

The value of i this day, or the mean distance of the points struck, is 89.7.



74. Wednesday,

74. *Wednesday, August 11, 1784; from 10 till 2.*

The air was warm, close, and thick.

Barometer 30.25; Thermometer 65° at 10 A. M.

N ^o	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		diam	wt		gun	pend			p	g	n	
	oz	inches	oz	dr			inches	inch	lb	inches		feet
1	4				D 65							
2	10	1.97	16	15	272	183	89.4	7	686.7	76.87	40.25	1561
3	8	.	.	.	240	176	89.7	6	688.0	76.89	40.24	1499
4	12	.	.	.	297	183	89.5	6	689.3	76.91	40.24	1566
5	10	.	.	.	258	168	88.8	7	690.6	76.93	40.24	1452
6	10	.	.	.	263	168	88.1	6	691.8	76.95	40.24	1466
7	12	.	.	.	293	177	88.2	8	693.1	76.97	40.24	1546
8	14	.	.	.	327	182	89.8	10	694.3	76.99	40.24	1565
9	14	.	.	.	310	175	89.8	9	695.5	77.01	40.24	1508
10	8	.	.	.	240	172	88.6	8	696.9	77.03	40.23	1505
11	8	.	.	.	232	164	89.7	8	698.1	77.05	40.23	1421
12	6	1.96	16	14		151	89.5	6	699.4	77.07	40.23	1319
13	6	.	.	.	200	157	88.6	6	700.7	77.09	40.23	1388
14	12	.	.	.	283	157	85.1	9	702.0	77.11	40.23	1448
15	14	.	.	.	318	169	88.0	7	703.2	77.13	40.23	1510
16	6	.	.	.	202	149	84.0	8	704.5	77.15	40.23	1398

The GUN was n^o 1, weighing, with the usual leads, 917 lb.

The PENDULUM as left hanging since yesterday.

The mean value of *i*, for the last two days, is 88.91.

After the experiments were ended this day, the pendulum was weighed, and found to be 706 lb. Now the original weight, when weighed at first on the 26th of July, seemingly with as much care as now, was 627 lb; to this add 61½ lb weight of balls and plugs lodged in it, and 10 lb of iron added on the 8th of August, and they make together 698½ lb; from this take 1.6 lb, for the diminution of the leaden facing of

of the pendulum, by the balls striking and piercing it, and there will remain only 697 lb, which the pendulum ought to weigh, and which is 9 lb less than it is actually found to weigh. I cannot imagine any cause to which this difference of weight may be attributed, as it is contrary to the effect heretofore experienced, the pendulum having always been found to lose in weight by hanging up; unless it arise from the moisture imbibed by the block in the 17 days it was up, and during all or the most part of which time it was very rainy weather, and the pendulum hung uncovered. And the probability of this will be heightened by considering that the block had lain by all the preceding winter, and till after midsummer this year, under cover, in the carpenter's shop, a circumstance which would make it very dry, and so render it apt to imbibe moisture from the continually foggy atmosphere and rain which have taken place ever since it was exposed. This increase of weight then, being 9 lb in the 17 days, or nearly half a pound per day, I have thought it safest to divide equally among all the days, by adding half a pound for each day it hung up, from the beginning of this year to the end of this day's experiments.

The object of this course was again to search out the maximum of the gun's charge; but it is not a good set of experiments, the velocities being not regular, perhaps owing to the bad state of the pendulum, which was very much shattered. However it sufficiently appears that there is but little difference among the velocities due to 8, 10, 12, and 14 ounces of powder.

Weight of Powder -	6 oz	8 oz	10 oz	12 oz	14 oz
Mean height of charge	4.4	5.7	7.0	8.1	9.5
Mean recoil of gun	201	237	264	291	318
Velocities by the pen- dulum - - -	$\left\{ \begin{array}{l} 1319 \\ 1388 \\ 1398 \end{array} \right.$	$\left\{ \begin{array}{l} 1499 \\ 1505 \\ 1421 \end{array} \right.$	$\left\{ \begin{array}{l} 1561 \\ 1452 \\ 1466 \end{array} \right.$	$\left\{ \begin{array}{l} 1566 \\ 1546 \\ 1448 \end{array} \right.$	$\left\{ \begin{array}{l} 1565 \\ 1508 \\ 1510 \end{array} \right.$
Mean ditto - - -	1368	1475	1493	1520	1528

75. Thursday,

75. *Thursday, September 9, 1784.*

Since the last experiments, the steadying-rods of the gun-frame having been lengthened, and the pendulum block repaired with a new core, &c. we attended to weigh and measure the several parts; the circumstances of which were as follows :

Weight of the pendulum	-	638	=	p
Theref. to its center of gravity		75.93	=	g
And its vibrations per minute		40.30	=	n

The new stay-rods of the gun-frame weigh 17 lb more than the old ones, so that now

	lb
The weight of iron in the frame is	205
Weight of gun and iron together	495
Weight of gun, iron, and leads -	934 = G

By this additional 17 lb weight of iron, the values of g and n , or the center of gravity and number of oscillations, will be altered; which will cause an alteration in our theorem $v = \frac{59}{96} \times \frac{Gg^c}{bin}$, by which the velocity of the ball is determined from the recoil of the gun, in Art. 36. The values of those two letters were, at Art. 42 and 43, found to be $g = 80.47$, and $n = 40.0$ for the gun n° 2; but the former will now become something less, and the latter something greater.

Now the old and new iron stay-rods were nearly of equal thickness. But the old rods extended only 29 inches, and the new ones 58 inches below the axis; the difference is 29; and the half difference, or $14\frac{1}{2}$ added to the old length 29, gives $43\frac{1}{2}$ inches below the axis, where the middle

middle or center of gravity of the additional length is situated, the weight of which part is 17 lb. But the center of gravity was found to be 80.47 below the axis, when the whole weight was 917 lb. Here, the difference of the two distances, or the distance between the two weights 17 and 917, being 37 inches, and the sum of the weights 934, we shall have $934 : 17 :: 37 : 0.67$ the change of the distance of the center of gravity; which being subtracted from 80.47, leaves 79.8 for the distance of the new compound center of gravity.

Also the correction of the value of n will be determined by the usual

formula $\frac{bin(inn - 140850)}{281700pg + bi(inn + 140850)}$, in which

$b = 17$, $i = 43.5$, $n = 40.0$, $p = 917$, and $g = 80.47$; which values, being used in that formula, give 0.1 for the correction of n ; to which add 40.0, and we shall have 40.1 for the new value of n , or number of oscillations per minute, for the gun n° 2; and consequently 40.2 for n° 1, and 40.0 for n° 3, and 39.9 for n° 4. Hence then the new values for the gun n° 2 are thus:

G	g	n	i	r
934	79.8	40.1	89.15	1000

Then, using these values of G , g , n , i , r , in the formula

$$v = \frac{59000}{96} \times \frac{Ggc}{birn} \text{ above-mentioned, it becomes}$$

$$v = \frac{205}{16} \times \frac{c}{b} \text{ for the velocity by the recoil of the gun;}$$

where b is the weight of the ball, and c the difference between the chords of recoil with and without a ball.

And when $b = 1.047 \text{ lb} = 16 \text{ oz } 12 \text{ dr}$, the same theorem is $v = 12\frac{1}{3}c$ for the gun n° 2. And every $\frac{1}{2}$ dram in the value of b will alter this theorem by the $\frac{1}{323}$ th part nearly.

Also for the gun n° 1 the above velocity must be decreased by the 400th part, and for n° 3 increased by the 400th part, and for n° 4 increased by the 200th part.

76. Friday,

76. Friday, September 10, 1784; from 10 till 1.

The weather fair; but not warm.

N ^o	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		diam	wt		gun	pend			<i>p</i>	<i>g</i>	<i>n</i>	
	oz	inches	oz	dr			inches	inch	lb	inches		feet
1	4				115							
2	4				116							
3	6				194							
4	6				190							
5	6				193							
6	4	1.96	16	12	143	143	88.0	8	638.0	75.93	40.30	1148
7	4	.	.	.	322	140	88.3	7	639.2	75.95	40.30	1123
8	4	.	.	.	324	144	89.3	7	640.3	75.97	40.30	1144
9	4	.	.	.	318	138	88.4	6	641.5	75.99	40.30	1110
10	6	.	.	.	433	173	88.5	7	642.7	76.02	40.30	1393
11	6	.	.	.	432	173	89.9	6	643.8	76.04	40.30	1374
12	6	.	.	.	430	172	90.1	7	645.0	76.06	40.30	1366
13	6	.	.	.	427	168	89.6	6	646.1	76.09	40.30	1345
14	8	.	.	.	519	188	90.0	7	647.3	76.11	40.30	1501
15	8	.	.	.	498	172	88.9	8	648.5	76.13	40.30	1394 D
16	8	.	.	.	529	190	89.6	6	649.6	76.15	40.30	1530
17	2	.	.	.		92	89.4	3	650.8	76.17	40.29	744
18	2	.	.	.	197	98	90.3	3	652.0	76.19	40.29	786
19	2	.	.	.	187	91	89.8	3	653.1	76.21	40.29	736

The GUN, n^o 1, without the leaden weights, weighed 495 lb.

The PENDULUM as specified the last day.

The plugs weigh 6 ounces to 7 inches long, not being of so dry wood as before. And this rate of the weight of the plugs to be continued till an alteration is announced.

The mean value of *i*, or point struck, is 89.29.

Here 439 lb weight of lead being taken off, at the distance 90.3 below the axis; and the center of gravity yesterday being at 79.8 distance,

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when the whole weight was 934 lb; therefore $495 : 439 :: 10.5 : 9.3$ the change of the center of gravity; and consequently $79.8 - 9.3 = 70.5 = g$ is the distance of the new center of gravity for this day.

Also the new number of oscillations per minute for this day will be found by this formula

$$375.3 \sqrt{\frac{gg-bi}{gg-bii}} = 375.3 \sqrt{\frac{gg-bi}{gg \cdot \frac{140850}{nn} - bii}} = 40.5;$$

where the values of the letters are thus, namely:

$$G = 934$$

$$g = 79.8$$

$$b = 439$$

$$i = 90.3$$

$$n = 40.2$$

Now in this day's experiments, the

Charge or weight of Powder	-	2 oz	4 oz	6 oz	8 oz
Mean height of ditto	- -	1.8	3.1	4.3	5.8
Mean recoil with ball	-	192	321	431	515
Ditto without	- - - - -	.	115	192	.
Difference, or c =	- -	.	206	239	.
Hence velocity by recoil	-	.	1170	1358	.
Mean ditto by the pendulum	755	1131	1370	1475	
Difference,	- - - - -		+ 39	- 12	
Or nearly the part	- - -		$\frac{1}{30}$	$\frac{1}{114}$	

These velocities from the recoil are found by the theorem

$$\frac{59}{96} \times \frac{ggc}{bin}, \text{ where the values of the letters are thus:}$$

$$G = 495$$

$$g = 70.5$$

$$b = 1.047$$

$$i = 89.15$$

$$n = 40.5$$

77. *Saturday, September 11, 1784; from 10 till 1.*

Very hot and clear weather.

N ^o	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		diam	wt		gun	pend			<i>p</i>	<i>g</i>	<i>n</i>	
	oz	inches	oz	dr			inches	inch	lb	inches		feet
1	4				58							
2	8	1.97	16	14	249 D	225	89.6	10	654.3	76.23	40.29	1814
3	8	1.92	16	4	248	206	89.6	6	655.4	76.25	40.29	1730
4	8	1.87	15	2	241	185	88.7	8	656.6	76.27	40.29	1693
5	8	1.97	16	14	262	224	89.7	3	657.8	76.29	40.29	1815
6	8	1.92	16	2	249	201	88.9	7	658.9	76.32	40.28	1725
7	8	1.87	15	2	236	177	88.6	7	660.1	76.34	40.28	1631
8	4	1.97	16	14	165	165	90.0	7	661.3	76.36	40.28	1341
9	4	1.92	16	2	155	146	89.3	6	662.4	76.39	40.28	1255
10	4	1.87	15	2	149	134	89.7	7	663.6	76.41	40.28	1228
11	4	1.97	16	14	165	165	89.9	6	664.8	76.43	40.28	1351
12	4	1.92	16	1	153	142	89.3	5	665.9	76.45	40.27	1233
13	4	1.87	15	2	146	132	89.3	5	667.1	76.48	40.27	1222

The GUN was n^o 3, and weighed 934 lb.

At n^o 2 the recoil 249 of the gun is too small; owing to the stylette, which ought to trace the arc, not marking all the way.

The PENDULUM as left yesterday.

The mean value of *i*, or point struck these two days, is 89.34.

The object this day was the effect of different sizes and weights of balls, and different degrees of windage.

The mean weight of balls and velocity, for the two weights of powder 4 and 8 ounces, are as follow:

D D 2

Powder's

Powder's wt ht		Ball's		Recoil gun	Veloc. ball
		wt oz dr	diam inches		
4	3'4	15 2	1'87	148	1225
.	.	16 2	1'92	154	1244
.	.	16 14	1'97	165	1346
8	5'9	15 2	1'87	239	1662
.	.	16 3	1'92	249	1728
.	.	16 14	1'97	262	1815

Here the decrease of the velocity is uniformly observable with the decrease of weight in the ball, and that in a very considerable degree, instead of increasing, which it ought to do, if the windage were the same, or the balls had the same diameter, and that in the reciprocal sub-duplicate ratio of the weight of the ball. Now that ratio is the ratio of $\sqrt{15\frac{1}{8}}$ to $\sqrt{16\frac{7}{8}}$, or of 11 to $11\frac{7}{11}$. Therefore as $11 : 11\frac{7}{11} :: 1346 : 1424$ the velocity the least ball would have had, if its diameter had been equal to the heaviest. But its velocity was actually no more than 1225; and therefore the difference 199, or $\frac{1}{7}$ of the whole, or $\frac{1}{6}$ of the experimented velocity, is the velocity lost by the difference of windage; although this difference was only $\frac{1}{16}$ of an inch, or $\frac{1}{20}$ of the caliber, which is no more than the usual windage allowed in service. But the force, or inflamed powder, lost by the same cause, will be $\frac{2}{7}$, or a double part of the velocity, because the velocity is as the square of the force or quantity of powder. Hence then, in charges with 4 ounces of powder, and a windage of $\frac{1}{20}$ of the caliber, $\frac{2}{7}$ of the charge is lost, or nearly a mean between $\frac{1}{3}$ and $\frac{1}{4}$.

And if the computation be made in like manner for the above charges of 8 ounces of powder, it will be found that the part of the charge lost by the same windage, will be, in the case of 8 ounces, $\frac{4}{13}$ of the whole; which is still more than the $\frac{1}{4}$ part, though somewhat less than in the case of 4 ounces. The reason of which is, that the ball is sooner out of the gun with the 8 oz charge, and so the fluid has less time to escape in.

78. Thursday,

78. *Thursday, September 16, 1784.*

To try the effect of firing the charge of powder in different parts of it.

N ^o	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		diam	wt		gun	pend			<i>p</i>	<i>g</i>	<i>n</i>	
		inches	oz	dr								
1	4	1.96	16	9	347	160	88.0	4	668.3	76.50	40.27	1373
2	4	.	.	.	348	161	87.8	7	669.4	76.52	40.27	1387
3	4	.	.	.	353	165	88.5	6	670.6	76.54	40.27	1413
4	4	.	.	.	350	161	87.5	6	671.8	76.57	40.27	1398
5	4	.	.	.	346	157	87.3	5	672.9	76.59	40.27	1369
6	4	.	.	.	352	159	87.2	5	674.0	76.61	40.27	1390

The GUN was n^o 3; its weight 500lb.

The PENDULUM as left yesterday.

The mean value of *i*, or point struck these 3 days, 89.03.

Powder		Recoil		Mean veloc.	
wt	ht	gun		of the ball	
4	3.1	-	-	34.9	- - 1388

The cartridge of n^o 1, 2, and 4 was fired at the fore part; n^o 3 and 5 behind; and n^o 6 in the middle: but there does not appear to be any difference among them.

79. *Tuesday,*

79. *Tuesday, September 21, 1784; from 10½ till 1½.*

The weather moderately warm.

N ^o	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		diam	wt		gun	pend			p	g	n	
	oz	inches	oz	dr			inches	inch	lb	inches		feet
1	4	1.97	16	12	166							
2	4	.	.	.	457	132	89.7	5	683.0	76.66	40.28	1125
3	4	.	.	.	451	132	91.3	4	684.1	76.68	40.28	1107
4	4	.	.	.	458	136	91.5	4	685.2	76.70	40.28	1140
5	6	.	.	.		157	88.9	4	686.3	76.72	40.28	1357
6	6	.	.	.	613	162	91.0	4	687.5	76.74	40.28	1371
7	6	.	.	.	591	153	90.1	4	688.6	76.76	40.28	1310 D
8	6	.	.	.	617	162	90.2	5	689.7	76.78	40.28	1388
9	8	.	16	10		163	89.6	5	690.8	76.80	40.27	1420
10	8	1.96	16	12		168	88.6	4	691.9	76.82	40.27	1471

The GUN was n^o 1, without any of the leaden weights. The gun itself now weighs only 179 lb, as it has been lightened 111 lb, by turning it down, to try if the velocity of the ball would be any less by making the gun lighter: but no difference appears, as the iron work is 205, the gun and iron together this day weigh 384 lb.

80. The PENDULUM as left yesterday, except that it had received a strengthening strap of iron, weighing 7 lb 13½ oz, which, reduced to its center of gravity, is placed at 79 inches below the axis. With this strap it weighed this morning, before the experiments commenced, 683 lb; which is 6.2 lb less than it ought to be by adding all the balls and plugs to the first weight; of which 6.2 lb difference, about 1.6 lb is for waste of the leaden facing, and the rest 4.6 lb is probably by evaporation: and as the time the pendulum has hung up is 11 days, the rate of

of evaporation is about $\frac{3}{7}$ of a pound per day. The 6.2 lb loss is divided equally among all the 32 experiments that have been made.

On account of the iron strap of 7.8 lb added at 79 inches, as above, the formula last given, for the variation in the center of gravity and number of oscillations, will need correction, namely the formula

$$G = 76.54 + \frac{i - 76.54}{670 + b} b,$$

$$N = 40.27 - \frac{i - 86}{217947} bi.$$

Now these formulæ, by making $i = 79$, and $b = 7.8$, become

$$G = 76.54 + .03 = 76.57$$

$$\text{and } N = 40.27 + .02 = 40.29$$

And hence the corresponding radical values are nearly

p	g	n
678	76.57	40.29

Which values, being substituted in the two general theorems, viz.

$$G = g + \frac{i - g}{p + b} b, \text{ and}$$

$$N = n - \frac{bin(inn - 140850)}{281700pg + bi(inn + 140850)},$$

they become

$$G = 76.57 + \frac{i - 76.57}{678 + b} b, \text{ and}$$

$$N = 40.29 - \frac{40.29bi(i - 85.92)}{8920300 + bi(i + 85.92)}$$

$$\text{or } N = 40.29 - \frac{i - 85.92}{220309} bi \text{ nearly:}$$

which are the new theorems hereafter to be used.

Note, the mean value of i , for the point struck the four last days, is 88.82; which, used in these last formula, give the corrected values of g and n , as inserted in their proper columns in the table of this day's experiments.—N° 7 is doubtful, and therefore omitted.

The

The means of this day are as below :

Powder				Recoil		Veloc. of	
wt	ht			gun		the ball	
4	3'0	-	-	455	-	-	1124
6	4'3	-	-	615	-	-	1372
8	5'5	-	-		-	-	1445

81. *Saturday, September 25, 1784.*

This day Major Blomfield alone tried some cartridges, of 8 oz each, by firing them behind, before, and in the middle ; but he found no sensible difference in the velocities.

He also discharged several low balls, weighing only 13 oz 3 dr, and having about .15 of an inch windage ; and the same balls, when covered with leather, so as to fit closely in the bore : but the velocities were the same ; probably owing to the fired powder quickly blowing off the leather.

The weight of the pendulum was increased 10 or 11 lb, namely, by 8 balls and 58 inches of plugs.

82. *Monday,*

82. Monday, October 4, 1784; from 11 till 2.

The weather dry, but cold and windy.

N ^o	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball feet
		diam	wt		gun	pend			p	g	n	
	oz	inches	oz	dr			inches	inch	lb	inches		
1	4				D149							
2	4				163							
3	4				164							
4	8	1.96	16	10	740	158	87.6	6	704.0	77.03	40.26	1440
5	8	.	.	14	742	166	88.2	6	705.0	77.05	40.26	1482
6	8	.	.	10	742	163	88.1	7	706.1	77.07	40.26	1482
7	8	.	.	14	729	158	87.6	6	707.1	77.09	40.25	1425
8	8	.	.	10	749	168	89.0	6	708.2	77.11	40.25	1517
9	8	.	.	14	751	166	88.8	6	709.2	77.13	40.25	1482
10	6	.	.	10	615	151	89.1	6	710.3	77.14	40.25	1367
11	6	.	.	14	604	150	90.2	7	711.3	77.16	40.25	1323
12	6	.	.	10	586	147	92.3	9	712.4	77.18	40.24	1289
13	6	.	.	14	610	152	91.9	8	713.4	77.20	40.24	1321
14	4	.	.	10	457	128	92.5	7	714.5	77.22	40.24	1124
15	4	.	.	14	453	120	92.4	8	715.5	77.24	40.24	1041
16	4	.	.	10	447	120	92.3	6	716.5	77.26	40.24	1059
17	2	.	.	14	270	85	91.5	5	717.6	77.27	40.23	747
18	2	.	.	10	271	85	91.2	4	718.6	77.29	40.23	762
19	2	.	.	14	279	87	91.4	4	719.7	77.31	40.23	768
20	4	.	.	14	459	128	92.3	5	720.7	77.33	40.23	1120

The GUN n^o 1, without the leads, weighed 384 lb.

The PENDULUM the same as left hanging since the last day.

This day was a continuation of the experiments with the light gun, again to try if the velocity was altered. But without effect. The means as below :

Powder's Weight	2 oz	4 oz	6 oz	8 oz
- - - height	1.73	2.94	4.12	5.42
Recoil of gun	273	454	604	742
Velocity of ball	759	1086	1325	1472

The mean weight of the balls is 16 oz 12 dr.

The mean value of i , for the point struck, was 89.3.

83. *Tuesday, October 5, 1784; from 11 till 2.*

The weather fine and warm.

N ^o	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		diam	wt		gun	pend			p	g	n	
	oz	inches	oz	dr			inches	inch	lb	inches		feet
1	4				48							
2	8	1.96	16	10	206	152	89.0	5	721.9	77.35	40.23	1404
3	8	.	.	14	213	158	87.6	5	723.0	77.37	40.23	1463
4	8	.	.	14	208	163	91.8	7	724.1	77.39	40.23	1443
5	8	.	.	14	$\frac{1}{2}$	158	91.0	6	725.2	77.41	40.22	1414
6	8	.	.	14	$\frac{1}{4}$	152	90.7	6	726.4	77.43	40.22	1367
7	8	.	.	13	$\frac{1}{4}$	153	91.3	6	727.5	77.45	40.22	1374
mean												1411

The GUN was n^o 1, with 687 lb of lead fixed to it, namely, 433 $\frac{1}{2}$ lb about the trunnions, and 253 $\frac{1}{2}$ lb lashed upon the upper side of the gun, close to, and before and behind the stem: these, with 384 lb for the gun and iron together, make in all 1071 lb.

The object was again to try if the velocity of the ball would be increased by diminishing the recoil of the gun. And for the severer trial, a great quantity of heavy timber was laid behind and against the cascable of the gun in the last three rounds, so as to stop the recoil intirely, which it did, excepting for about the $\frac{1}{2}$ or $\frac{1}{4}$ of an inch, which the gun pushed the timber back, as expressed in the column of recoil. But the result is still the same.

The PENDULUM the same as left hanging since yesterday.

The mean value of i , or point struck the last 6 days, is 89.4.

84. *Wednesday, October 6, 1784.*

The weather clear, but windy.

N ^o	Powder oz	Ball's			Vibration of		Point struck inches	Plugs inch	Values of			Veloc. ball feet
		diam inches	wt oz	dr	gun	pend.			<i>p</i> lb	<i>g</i> inches	<i>n</i>	
1	4				62							
2	8				144							
3	.				144							
4	.				148							
5	.				147							
6	.				155							
7	.				157							
8	.	1.95	16	9	272	150	88.4	9	728.7	77.46	40.22	1416
9	.	.	.	.	268	146	88.8	7	729.8	77.48	40.22	1375
10	.	.	.	.	279	150	88.1	5	730.9	77.49	40.22	1426
11	.	.	.	.	279	147	88.7	7	732.0	77.51	40.21	1390
12	.	.	.	.	273	150	87.4	6	733.2	77.52	40.21	1442
13	.	.	.	.	282	174 ^D	87.4	8	693.4	76.83	40.27	1566 ^D
mean												1436

The GUN n^o 1, with leads, weighed 817 lb.

The PENDULUM as left yesterday.

The mean value of *i*, or point struck, these last 7 days, 89.3.

The object this day was the effect of cork wads, and of different degrees of ramming. The cork wads were near an inch long, and were made to fit very tight, being rather more than 2 inches diameter; and weighed 5 drams each.

N^{os} 1, 2, 3, 8, 9 were without wads,
 4, 5, 10, 11 with a wad gently pressed home,
 6, 7, 12, 13 with a wad, and hard rammed by 2 men.

At n^o 12 one of the iron bands of the pendulum broke, and fell across the measuring arch. The band weighed 41 lb, and n^o 13 was fired after the band was removed, and consequently 41 lb must be deducted.

The velocities are

1416	}	1396 the mean without wads
1375		
1426	}	1408 with wads not pressed.
1390		
1442	}	1442 with wads very hard rammed.
D		
148		Mean recoil without ball
275		Ditto with ball.

N^o 13 is very doubtful, the vibration of the pendulum being evidently too large; perhaps 174 had been set down instead of 164.

In the above there seems to be some small advantage in favour of the wads. But I suspect the difference is only accidental; and the number of experiments is too small to afford any tolerably good mediums.

85. Monday,

85. Monday, October 11, 1784; from 11 to 2.

The weather cold and cloudy.

N ^o	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		diam	wt		gun	pend			p	g	n	
	oz	inches	oz	dr			inches	inch	lb	inches		feet
1	4				63							
2	8				143							
3	.	1.95	16	9		141	87.4	10	733.5	77.52	40.21	1357
4	.	.	.	9	285	160	85.9	9	734.9	77.54	.	1569 D
5	.	.	.	8	275	150	86.8	7	736.4	77.56	.	1465
6	.	.	.	8	267	149	86.1	8	737.8	77.58	.	1470
7	.	.	.	8	268	145	86.2	8	739.3	77.60	.	1433
8	.	.	.	8	264	146	87.0	9	740.7	77.62	.	1432
9	.	.	.	8	274	147	85.4	6	742.2	77.64	.	1472
10	.	.	.	8	272	148	86.5	7	743.6	77.66	.	1467
11	.	.	.	5	267	142	85.9	5	745.1	77.68	.	1436
12	.	.	.	5	261	137	86.5	9	746.5	77.70	.	1379
13	.	.	.	5	263	137	85.9	7	748.0	77.72	.	1392
14	.	.	.	5	269	143	85.7	6	749.4	77.74	.	1459
mean												1444

The GUN n^o 1, weighed 817 lb.

The PENDULUM had had its band repaired, which did not however alter its weight. The whole weighed this morning 733½ lb. Now the weight of the balls and plugs in the last 5 days is 62 lb, which, being added to 683 lb, the weight of the pendulum on September 21, it makes 745 lb, which is 11½ lb more than it weighed this morning. For this defect I know of no cause but evaporation: for in this time there was no waste of leaden facing, as the other end of the block was used, which was not covered with lead. The time in which this 11½ lb was lost is 20 days, which is nearly at the rate of ½ a pound each day. This defect is therefore divided equally among all the days.

The

The mean value of i , for the point struck these 8 days, is 88.8.

The object this day was again the effect of cork wads, and different degrees of ramming.

		Ht. of Powder	Mean Veloc.
N ^{os}	2, 9 were without wads - - -	5.85	1472
	3, 5, 7 with wads, not rammed -	5.87	1418
	6, 8 a wad, and very hard rammed	4.40	1451
	10, 11, 12 a wad, and moderately rammed	5.20	1427
	13, 14 2 wads over powder and 1 over ball, and very hard rammed }	4.45	1426
Mean of all - - - - -		5.15	1444
		Wt. of ball	Mean Veloc.
N ^{os}	3, 4 - - - - -	16 9	1463
	5, 6, 7, 8, 9, 10 - - - - -	16 8	1456
	11, 12, 13, 14 - - - - -	16 5	1417
Mean of all - - - - -		16 7	1444

In this course the wads have no perceptible effect.

86. Tuesday,

86. Tuesday, October 12, 1784; from 11 till 1.

The weather fine and clear.

N ^o	Powder	Ball's			Vibration of		Point struck	Plugs	Values of			Veloc. ball
		diam	wt		gun	pend			<i>p</i>	<i>g</i>	<i>n</i>	
	oz	inches	oz	dr			inches	inch	lb	inches		feet
1	8				123							
2	16	1.96	16	11		208	87.1	9	750.9	77.76	40.21	2047
3	16	.	.	.	425	219	86.0	9	752.3	77.78	40.21	2187
4	16	.	.	.	409	200	85.6	10	753.8	77.79	40.20	2011
5	16	.	.	.		152	87.1	7	755.2	77.81	40.20	1505 D
6	16	.	.	.	389	197	85.4	11	756.7	77.83	40.20	1994
7	16	.	.	.	412	204	85.7	8	758.1	77.85	40.20	2062
8	16	.	.	.	408	203	85.5	8	759.5	77.86	40.20	2061
mean												2060

The GUN n^o 4, weighed 928 lb.

The PENDULUM as left yesterday. But it was quite broken and useless at the end of these experiments.

The mean value of *i*, or point struck these 9 days, 88.6.

The object this day was the effect of firing the charge in different parts, either before, or behind, or in the middle : for which the means are as below :

	Mean Veloc.
N ^{os} 2, 6 fired before	- 2020
3, 7 in the middle	- 2124
4, 8 behind	- - - 2036
Mean of all	- - - 2060
Mean recoil of gun	- - 409
N ^o 5 is omitted as doubtful.	

The end of Experiments in 1784.

EXPERI-

EXPERIMENTS IN 1785.

37. **S**EVERAL of the experiments of the two former years being not so regular as might be wished, we have again undertaken to repeat some of them, and to add still more to the stock already obtained, that the mediums upon the whole may be tolerably exact, the great number of repetitions counteracting the unavoidable small irregularities, and deviations from the truth, in experiments instituted upon so large a scale. For this purpose we begin with the gun n° 2, and use charges of 8 ounces of powder; and have formed the resolution of firing every shot into a fresh and sound part of the block of wood, and changing the block very frequently, before it become too much battered, that the penetration of the ball and the force of the blow may be obtained with the greater degree of accuracy.

It is also proposed to procure some good ranges, to compare them with the initial velocities made under the same circumstances; from the comparison of which we may estimate the effects of the resistance of the air, and so lay a foundation for a new theory of gunnery. It is rather difficult to obtain with accuracy such long ranges as our initial velocities would produce, being from 1 mile to 2 miles, when the projection is made at an angle of 45 degrees; for in such long ranges our small balls cannot be seen, when they fall to the ground. We were obliged therefore to have recourse to the water, in which the fall of the ball can be much better perceived; because the plunge of the ball in the water, breaking the surface and throwing it up, makes the place visible at a great distance. But then another difficulty occurs, how to obtain exactly the distance of the fall, or length of the range, as the mark
made

made in the surface of the water is visible but for a moment. This difficulty however, our situation at Woolwich, close by the river Thames, enabled us to overcome, as well as afforded us a good length of range. For at our situation in the Warren, the river makes a remarkable turn, and forms below us the part called the Gallions Reach, a map of which is here given in plate iv. In this map, A denotes the point where the guns were placed, being the Convicts' Wharf, which is so called because it is there that the convicts, or felons condemned to work on the river Thames, land their gravel, and upon which they usually labour. From this point we have a convenient range of about a mile and a half towards B, in the county of Essex, where there is a private or merchant's powder magazine. The buildings near C consist of the academy and a noble range of store-houses; and from this point we should have had a still longer and more convenient range, had not our view from hence been interrupted by four large hulks, which lie, for the use of the convicts, in the river opposite the part between this point and the point A. Having found this convenient situation for our operations, we made an exact survey and map of the two sides of the river, both ways beyond the extent of the ranges; and fixed on convenient stations at D and E on the south side, and F and G on the north side of the river, to place two parties of observers, who might mark the place where each ball should fall in the water, as well as note down the time employed by the ball in flying through the air, from the visible discharge of the gun to the plunge of the ball in the water. The method of determining the place of the fall was this: Two parties of observers, consisting of three or four steady and intelligent young gentlemen in each party, having taken their stands at D and F, or E and G, according to the expected length of the range, carefully watched the discharge of every ball from the gun at A; then tracing it, as it were, through the air by the loud whizzing noise it made in its flight, their eye was prepared and directed gradually towards the place of the fall, which they seldom missed of observing. Then immediately

on perceiving the plunge, some of them noted the time of flight, by a good stop watch, while others observed some remarkable land object on the opposite coast, and directly in a line with the place in the water where the ball fell. This done, they directed the telescope or sights of an instrument, such as a theodolite or plain table, to that object, and noted the position of it. This being done by each party of observers, and the line of position from each station drawn on the plan, afterwards at leisure, the intersection of those lines gave very exactly the place of the fall, and consequently its distance from the gun. In this manner then were determined all the ranges and times of flight registered in the following experiments; those places being left blank where the observation was either doubtful, or not made at all. The times of flight were also sometimes observed at the gun itself, where the plunge of the ball could often be perceived.

In this map of the river in plate iv, the dotted line on each side of the river denotes low-water mark; the first black line next without it denotes high-water mark; and the other, or outermost line, is the land bank which has been raised in former ages, from Greenwich for many miles below, and with immense labour, to prevent the waters of the river from overflowing the adjacent fields, which it would do every tide, as they lie low and are otherwise very marshy.

88. *Wednesday, August 31, 1785.*

EMPLOYED this day in making part of the survey by the side of the river, for forming the map, and fixing the stations proper for the parties of observers to occupy, in watching the fall of the balls in the river; and for other purposes.

We

We weighed and measured the pendulum, which had been prepared in a very complete manner, and with stronger bands than before. It weighed just 795 lb. And, by a mean of several times balancing and vibrating, we found $78\frac{1}{2}$ inches to be the distance of the center of gravity below the axis, and 40.07 the number of oscillations per minute.

After executing part of the survey by the side of the river, we fired a few balls upon the water, from the Convicts' or Proof Wharf, to try whereabouts they will fall, and thereby to judge of the proper places for the observers to be stationed at. The gun was n° 2, with 8 ounces of powder, and was tried at different elevations. When the gun was elevated at 45 degrees, the balls ranged much too far, going beyond the stretch of the river, and falling on the coast of Essex below the point B. But at 15 degrees elevation, the balls ranged to a very convenient distance, namely, a little more than a mile. And their fall in the water could be very well seen from the side of the river nearly opposite the place of the fall, and sometimes from the gun itself.

Upon this occasion I took out with me, and employed the first class of Gentlemen Cadets belonging to the Royal Military Academy, namely, Messieurs Bartlett, Rowley, De Butts, Bryce, Wm. Fenwick, Pilkington, Edridge, and Watkins, who have gone through the science of fluxions, and have applied it to several important considerations in natural philosophy. Those gentlemen I have voluntarily offered and undertaken to introduce to the practice of these interesting experiments, with the application of the theory of them, which they have before studied under my care. For, although it be not my academy duty, I am desirous of doing this for their benefit, and as much as possible to assist the eager and diligent studies of so learned and amiable a class of young gentlemen; who, as well as the whole body of students now in the upper academy, form the best set of young men I ever knew in my life; nay, I did not think it even possible, in our state of society in this country, for such a number of gentlemen to exist together in the constant daily

habits of so much regularity and good manners; their behaviour being indeed perfectly exemplary, and the pure effect of true philosophical principles, arising from a rational conviction of the propriety of a regular good conduct, which is such as would do honour to the purest and most perfect state of society that ever existed in the world: and I have no hesitation in predicting the great honour, and future services, which will doubtless be rendered to the state by such eminent instances of virtue and abilities.

89. *Thursday, September 1, 1785.*

WENT out again with the same class of eight young gentlemen, to complete the survey of the river side. The weather changed to rain after we were out, which continued the whole time, and to such a degree as to wet us intirely through all our clothes. Yet every one went through the business, not only willingly, but even chearfully.

90. *Friday,*

90. *Friday, September 2, 1785; from 9 till 3.*

The weather rather windy and cloudy.

Barometer 29.8; Thermometer within 66°.

Went with Major Blomfield and the same class of cadets, and made the following set of fourteen experiments, the first 8 balls being fired into the pendulum, and the other 6 down the river, to get the corresponding ranges.

N ^o	Powder	Ball's			Vibr. pend	Point struck	Plugs	Values of			Veloc. ball	Pene- tration
		wt		diam				<i>p</i>	<i>g</i>	<i>n</i>		
	oz	oz	dr	inches		inches	inch	lb	inches		feet	inches
1	8	16	13	1.965	128	81.0	10	795.0	78.33	40.07	1438	18.0
2	.	.	.	.	139	85.7	9	796.6	35	07	1479	19.9
3	.	.	.	.	141	90.3	11	798.3	38	07	1428	
4	.	.	.	.	146	92.7	12	800.0	40	07	1444	16.7
5	.	.	.	.	163	96.2	12	801.6	43	06	1557	20.3
6	.	.	.	.	171	94.0	10	803.2	45	06	1675	20.7
7	.	.	.	.	149	89.1	13	804.8	47	06	1543	16.4
8	.	.	.	.	144	91.5	11	806.4	50	06	1457	16.6
means											1503	18.9
9	.	.	.	.	Time sec 14	Range feet 6110						
10	.	.	.	.		6060						
11	.	.	.	.		not						
12	.	.	.	.		feen						
13	.	.	.	.		5760						
14	.	.	.	.		5735						
mean						5916						

The GUN was n^o 2. It was not hung on an axis, as in the two former years, but mounted on a small carronade carriage, made for the purpose,

purpose, both in the last 6 rounds, which were fired down the river, and in the first 8 rounds, which were fired into the new pendulum, at the same distance as formerly, or about 35 feet, and each ball into a fresh part of the wood, both to obtain the force of the blow the more accurately, and to take the penetration of the ball in the solid wood, which we did every time by pushing in a wire to touch the hinder part of the ball: these penetrations are various, according as the part struck was more or less compact, and they are rather larger than was expected, the medium of all being $18\frac{4}{16}$, although the block of elm, as the carpenters assured us, was found, dry, and well-seasoned wood. The penetrations are set down in the last column, and are for the fore part of the balls, the diameter having been always added to the length of the wire.

The POWDER was not of the same parcel as the two former years; but it was from the same maker, and made as nearly similar to the former as might be. The charges were gently set home, and all circumstances made alike. The mean length of the charge of 8 oz was 4.84.

The PENDULUM had been kept close covered with a painted canvas cloth since the first day that it was weighed and measured, to preserve it from the weather. The plugs weighed 9 ounces to every 11 inches in length; the whole weight of all the plugs, together with that of the 8 balls, make up 13 lb, wanting only an ounce and a half; and when the pendulum was taken down and weighed this afternoon, its weight was found to be 808 lb, which is just 13 lb more than its weight at first. So that it has neither lost weight by evaporation, nor gained by imbibing moisture: owing, probably, to the circumstance of being covered by the painted canvas.—All the apparatus was in good order, and the experiments all very accurately made.

At the beginning of these experiments, the values of p , g , n , being $p = 795$, $g = 78\frac{1}{3}$, $n = 40.07$; if these values be substituted in the two theorems

$$G = g + \frac{i-g}{p+b} b$$

$$N = n - \frac{bin(inn - 140850)}{281700pg + bi(inn + 140850)}$$

for the correction of g and n , they become

$$G = 78.33 + \frac{i-78.33}{p+b} b$$

$$N = 40.07 - \frac{40.07bi(i-87.16)}{10855700 + bi(i+87.16)}$$

$$\text{or } N = 40.07 - \frac{i-87.16}{270940} bi \text{ nearly.}$$



And by these theorems the numbers in the columns g and n are made out, the mean value of i , or point struck, being 90.1.

The last 6 rounds were fired down the river from the Convicts' or Proof Wharf at A, and the place of the fall observed by two parties of the cadets, stationed at D and E. The gun had 15 degrees elevation. The fall of the first only could be seen at the gun, where the time of flight was observed by a stop watch, and found to be 14 seconds. The two parties of observers at D and E had no time-piece with them, so that the other times of flight could not be observed. The medium range is 5916 feet or 1972 yards. The last two balls went close over the heads, and fell just beyond, the lower party of observers, at E; yet notwithstanding their imminent danger, they gallantly resolved to keep their ground, if any more rounds should be fired, not knowing immediately that we intended not firing any more at that time. These two rounds were probably deflected thus a little from their course by the usual causes of deviation. And perhaps the two former rounds had been still farther deflected, and thrown on the land, as the observers saw nothing of them. But the gun was pointed in a direction rather nearer this south side of the river.

91. Thursday,

91. *Thursday, September 8, 1785; from 12 to 3.*

The weather close and warm, rather hazy.

Barometer 30.02; Thermometer 65° within, but warmer without.

N ^o	Powder	Ball's			Time	Range	Whereabouts the Balls fell
		wt		diam			
	oz	oz	dr	inches	sec	feet	
1	8	16	12	1.96	14	6460	Near the middle of the river
2	.	.	.	.		6080	Near the north side
3	.	.	.	.			
4	.	.	.	.	15	6040	Ditto
5	.	.	.	.	15½	6540	Ditto
6	.	.	.	.	15	6460	Near the middle
7	.	.	.	.	14	5720	On the south bank, and within 40
				means	14.7	6216	yards of the lower station E

These 7 rounds were fired down the river from the same place as before; the elevation of the gun being 15 degrees, and all other circumstances the same as before. The gun was pointed nearly to the middle of the river; yet the balls fell mostly wide of the direction, and that both ways, some falling near one side of the river, and some near the other, though there was not the least wind. The times of flight were taken with a stop watch, at the lower station of observers at E, by noting the time between seeing the flash of the gun and the plunge of the ball in the water. They run from 14 to 15½ seconds, and accord very well with the ranges, the larger to the larger: the medium is 14.7 seconds; and the medium range 6216 feet, or 2072 yards. N^o 3 was not seen. The mean length of charge 4.8 inches.

The same parties of young gentlemen kept their station very gallantly, and make no hesitation in offering to attend and observe there for the remainder of the experiments, although some of the balls this day again fell near them, and one indeed within 40 yards of them.

92. *Friday,*

92. Friday, September 9, 1785; from 9½ till 1.

The weather very fine and warm.

Barometer 29.93; Thermometer, within, 66°.

N ^o	Powder	Ball's		Vibrat. pend	Point struck	Plugs	Values of			Veloc. ball	Penetr.
		wt	diam				<i>p</i>	<i>g</i>	<i>n</i>		
	oz	oz	dr		inches		lb	inches		feet	inches
1	4	16	12	99	79.0	9	805.3	78.48	40.06	1162	16.7
2	.	.	.	107	82.3	10	806.7	49	06	1208	16.4
3	.	.	.	114	87.1	10	808.1	50	07	1218	16.7
4	.	.	.	115	87.3	9	809.6	51	07	1228	
mediums										1204	16.6

We fired these 4 rounds into the same pendulum as we left hanging on September 2, which had been kept under cover since that time. After these 4 rounds, it weighed 811 lb, which is $2\frac{3}{4}$ lb less than it ought to be when the weight of the 4 balls and plugs are added to its former weight, and which $2\frac{3}{4}$ lb it must be supposed to have lost by evaporation in the course of the 7 days, which was mostly dry, warm weather.

The plugs weighed 9 oz to 14½ inches.

Mean length of the charge 3.0.

We could not venture to fire down the river this day, on account of the great number of ships that were upon it.

93. *Saturday, September 10, 1785; from 12 till 2.*

Fine dry weather.

Barometer 29.8; Thermometer 66°.

N ^o	Powder	Ball's			Penetr.	Recoil
		wt		diam		
	oz	oz	dr	inches	inches	inches
1	2	16	10	1.96	6.1	2.5
2	4	.	.	.	12.2	7.0
3	8	.	.	.	20.8	15.8
4	2	.	.	.	6.7	3.0
5	4	.	.	.	14.4	9.0
6	8	.	.	.	23.0	17.5
7	2	16	12	.	7.8	2.5
8	4	.	.	.	14.0	7.5
9	8	.	.	.	20.7	

These 9 balls were fired into the root end of a block of elm, laid upon the ground, to obtain the penetration with different charges, each ball being fired into a fresh and sound part of the wood, and in the direction of the fibres. The wood was moist within, as we discovered by boring out the balls; but it was hard and firm of its kind, being in the root, or the root end after the body of the tree was sawed off from it. The penetrations are for the fore part of the ball, as usual.

The gun was n^o 2, and mounted, as in all the experiments of this year, on a small sea gun carriage, without trucks, but fixed on a base like a mortar bed, and slid along the ground or platform in recoiling.

The muzzle was placed at 79 inches from the face of the block. The mean penetrations and recoils are as follows:

Powder

Powder	Penetr.	Recoil
2 OZ	6.9	2.7
4	13.5	7.8
8	21.2	16.7

So that the penetrations are nearly 7, 14, 21, or nearly as 1, 2, 3, or as the logarithms of the weights of powder.

94. *Wednesday, September 14, 1785; from 10 till 12½.*

A fine warm day.

Barometer 30.5; Thermometer, within, 67°.

N ^o	Powder	Ball's			Time	Range
		wt		diam		
	oz	oz	dr	inches	seconds	feet
1	4	16	12	1.96		
2	.	.	.	.		
3	.	.	.	.	10	4730
4	.	.	.	.	7 $\frac{1}{4}$	4030
5	.	.	.	.	8	4450
6	.	.	.	.		4380
7	.	.	.	.		
8	.	.	.	.		
mediums					8.4	4398

These 8 rounds were fired down the river as before. The gun n^o 2, and elevation 15 degrees, as usual. One party of the young gentlemen was stationed at D as before, but the other on the north side of the river at Deval's house at F. This last party saw only one ball plunge, and the first party saw four; which however proved sufficient for determining their ranges, because they all fell near the middle of the river, a circumstance which we also at the gun could sometimes perceive.

The mean time of flight is about $8\frac{1}{2}$ seconds, and the mean range 4398 feet, or 1466 yards.

95. *Saturday, September 17, 1785.*

N ^o	Powder	Ball's			Penetr.
		wt		diam	
	oz	oz	dr	inches	inches
1	12	16	12	1.96	22.0
2	14	.	.	.	23.6
3	16	.	.	.	24.0
4	10	16	14	1.97	22.3
5	8	.	.	.	18.1

These 5 balls were fired into the same block of elm root as on the 10th instant, to get a greater variety of penetrations.

96. *Tuesday, September 27, 1785.*

N ^o	Powder	Ball's			Penetr.	Part of the Charge fired at
		wt		diam		
	oz	oz	dr			
1	8	17	0		20.5	} Back part
2	.	.	.		20.6	
3	.	.	.		21.6	} Middle
4	.	.	.		20.5	
5	.	.	.		11.0	} Fore part
6	.	.	.		17.3	

These 6 also were fired, from the same gun, into the same block, to try the difference by firing the cartridge either behind, or before, or in the middle.—There must be some mistake in the numbers in the last two rounds, which cannot possibly differ so much from the other numbers.

97. *Wednesday,*

97. *Wednesday, September 28, 1785.*

A fine clear day.

Barometer 30.35; Thermometer 60.

N ^o	Powder	Ball's			Elevation degrees	Time seconds	Range feet
		wt		diam inches			
	oz	oz	dr				
1	4	16	10	1.97	15		5180
2	4	.	.	.	.	8 $\frac{3}{4}$	4370
3	4	.	.	.	.		
4	4	16	7	.	.	7 $\frac{3}{4}$	4020
5	2	.	.	.	45		5120
6	2	.	.	.	.	21 $\frac{1}{2}$	5300
7	2	.	.	.	.	21	5200
8	2	.	.	.	.		4120
9	4	16	11	1.95	15		
10	4	.	.	.	.	13 $\frac{1}{2}$	5770
11	2	16	7	1.97	45	23	5600
12	2	16	10	.	.		

These 12 rounds were fired down the river; the gun, stations, parties of cadets, &c. as before. The fall of those balls was not seen whose range is not set down. With 2 oz of powder the gun was elevated 45 degrees, but with 4 oz only 15 degrees, as before. The mediums are as below.

Powder	Elev.	Time	Range
2 oz	45°	22"	5068
4	15	8 $\frac{1}{4}$	4523

Rejecting n^o 10, as very doubtful; a mistake most likely having been made in the weight of the powder.

98. *Thursday,*

98. *Thursday, September 29, 1785.*

A fine clear day.

Barometer 30.35; Thermometer 60.

N ^o	Powder	Ball's			Elevation degrees	Time seconds	Range feet
		wt		diam inches			
1	2	16	12	1.95	45	20	5120
2	.	.	.	.	.	19	4730
3	.	.	.	.	.	20½	5370
4	.	.	.	.	.	20	5120
5	.	.	.	.	.	22½	5510
6	.	.	.	.	.	20	5050
7	12	.	.	.	15	17	7120
8	.	.	.	.	.	10 D	4860 D
9	.	.	.	.	.	9 D	4880 D
10	.	.	.	.	.	14	6660
11	.	.	.	.	.		5500
12	.	.	.	.	.		7520

These 12 rounds were fired on the river, and observed as before. N^{os} 8 and 9 are very doubtful: the means of the rest are as below.

Powder	Elevation	Time	Range
2 OZ	45°	20½	5150
12	15	15½	6700

99. The same day the following 6 rounds were fired into the block of elm root, to try the penetrations with and without wads; the first 4 being

being with a wad over the powder, and hard rammed; the other two without.

N ^o	Powder	Ball's			Penetration
		wt		diam	
	oz	oz	dr	inches	inches
1	8	15	12	1.95	16.1
2	.	.	.	.	21.4
3	.	.	.	.	20.6
4	.	.	.	.	19.8
5	.	.	.	.	19.8
6	.	.	.	.	21.0

} With wads

} Without wads

100. *Tuesday, October 4, 1785.*

Fine morning, but turned to rain about noon.

Barometer 29.93.

N ^o	Powder	Ball's			Time	Range
		wt		diam		
	oz	oz	dr	inches	seconds	feet
1	8	15	3	1.96		6330
2	.	.	.	.		5770
3	.	.	.	.		
4	.	.	.	.	8½	4800
5	.	.	.	.		4880
medium						5600

These 5 rounds were fired on the river, and observed as before.

The Gun was n^o 3, and its elevation 15 degrees.

The balls were not good ones, and the ranges very irregular; and the medium 5600 feet, or 1867 yards, too low; perhaps owing to the lightness of the balls.

101. *Tuesday,*

101. *Tuesday, October 11, 1785.*

The weather fine.

Barometer 29.88; Thermometer 60.

N ^o	Powder	Ball's			Time seconds	Range feet
		wt		diam		
	oz	oz	dr	inches		
1	8	15	12	1.96		5580
2	.	.	.	.		
3	.	.	.	.	10 $\frac{1}{4}$	5270
4	.	.	.	.		5990
5	.	.	.	.	9	4910
6	.	.	.	.	11	5750
7	.	.	.	.	9 $\frac{1}{2}$	6140
8	.	.	.	.	11	5700
means					10 $\frac{1}{7}$	5620

These 8 were fired in the river, and observed as before.

The GUN was n^o 3, and was elevated 15 degrees.

The ranges are again low, probably from the lightness of the balls.

The usual causes of deflection carried three of the balls, namely, the 1st, 7th, and 8th, very near the south station at E; and then fell almost close to the party there. In general it was observed that the balls deviate from their line of direction, or middle line of the river, to each side, by half the breadth of the river, or from 300 to 400 yards!

The end of Experiments in 1785.

A SUM-

EXPERIMENT IN 1786.

102. Monday, June 12, 1786; from 10 till 1.

Fine weather.

Barometer 29.89; Thermometer 63° at 9 A. M.

N ^o	Powder	Ball's			Elevat gun degrees	Time ft	Range
		diam inches	wt oz dr				
1	2	1.96	16 6	15			
2	.	1.96	. 6	.	14 D	5000 D	
3	.	1.96	. 6	.		5040 D	
4	.	1.96	. 6	.	8	3920	
5	.	1.97	. 7	.	7½	3560	
6	.	1.96	. 5	.	8½	3910	
7	.	1.96	. 5	.	10½	4450	
8	.	1.96	. 5	.	9½	4280	
9	.	1.95	. 5	.	8½	3910	
10	.	1.96	. 4	.	15 D	5600 D	
11	.	1.96	. 4	.	8½	3910	
12	.	1.96	. 4	.	11½	4750	
13	.	1.96	. 4	.	9½	4270	
14	.	1.96	. 4	.	10	4230	
15	.	1.96	. 4	.	9½	4000	
16	.	1.95	. 3	.		4960 D	
17	.	1.95	. 9	.	9½	4420	
18	4	1.95	. 3	.		4840	
19	4	1.96	. 3	.	11½	4690	
20	4	1.96	. 2	.	10½	5650	
medi- ums }	2	1.959	16 5	15	9¼	4130	
	4	1.957	16 2½	15	11	5060	

The GUN was n^o 2.

The ranges were taken from observations, as before, at Duval's house, and the first gibbet. The first 17 rounds were fired this year, with 2 ounces of powder, to complete the series of ranges at 15 degrees elevation of the gun; and the last three rounds, with 4 ounces, to try if the powder was of the same strength as before: and which, by comparing these three ranges with those of last year, appears to be now somewhat stronger. So that these ranges and times, it may be presumed, are too great in respect of those of last year. They are also evidently very irregular; owing perhaps to the inequalities of the balls, which were only the remaining outcasts from the whole stock we first began with, having been rejected either from their lightness, or from the irregularities of their surfaces. And sometimes indeed the ranges and times, here set down, were not very accurately obtained. The mediums of all, except those marked doubtful, are placed at the bottom.

A SUMMARY OF THE EXPERIMENTS : WITH PHILOSOPHICAL
REMARKS AND DEDUCTIONS.

103. **W**E have now got through this long three-years course of experiments ; and have detailed them in so minute and circumstantial a manner, as to enable every person fully to comprehend and make his own use of them ; without subjecting him to the dissatisfaction of having mediums and results forced upon him, unaccompanied with the fair and regular means of assuring himself both of their justice and propriety. We are now therefore to make some use of these experiments, by pointing out the philosophical laws and deductions that flow from them, and making such other remarks as may be suggested by the various circumstances of them, or that may be useful for improving or farther extending experiments attended with such important consequences in natural philosophy. And for these beneficial purposes, it will first be necessary to bring the mediums and results together into an abstract, or one comprehensive point of view ; to form as it were the sure and regular foundation for the future structure we hope to be able to raise upon them.

OF THE LENGTH OF THE CHARGE.

104. **AND** first we shall deduce the lengths or heights of the charge of powder, for every two ounces in weight ; or the part of the bore of the gun which every charge occupies : a thing very necessary both to shew the part of the bore, occupied by the charge, corresponding to the greatest or any other velocity of the ball, as also to compute *a priori*,

H H 2

from

from theory, the velocity due to every charge of powder. Now the length of the charge was taken at every experiment, by means of the divisions of inches and tenths marked on the rammer, and the mediums of most of them are specified for each day in the preceding account of the experiments; and those mediums of each day are here in the following table collected and ranged in columns, each under its respective weight at top, extending from 2 to 20 ounces:

2	4	6	8	10	12	14	16	18	20
1.7	3.2	4.6	5.9	6.93	8.4	9.8	10.6	12.3	13.2
1.7	3.2	4.4	5.9	6.97	8.2	9.27	11.0	12.15	13.3
2.1	3.3	4.5	6.1	6.9	8.3	9.55	11.2		13.2
1.8	3.23	4.43	6.2	6.97	8.23	9.4	11.4		
1.8	3.33	4.2	5.67	7.0	8.3	9.53	10.9		
1.7	3.24	4.5	6.0		8.37		11.0		
1.7	3.2	4.44	5.7		8.07		10.8		
1.87	3.2	4.4	5.72				11.13		
1.9	3.1	4.37	5.6				11.38		
1.85	3.17	4.27	5.63				11.26		
1.9	2.9	4.28	5.65				11.1		
1.8	3.4	4.12	5.6				10.6		
1.9	3.13		5.6				10.97		
1.9	3.1		5.83				10.87		
1.83	3.08		5.7				10.85		
1.73	3.4		5.77				10.79		
	3.1		5.88						
	3.0		5.45						
	2.95		5.4						
	3.0		5.74						
	3.1		5.85						
	3.1		5.4						
	3.03		4.84						
			4.8						
1.82	3.15	4.38	5.66	6.95	8.27	9.51	10.99	12.22	13.23

and in the lowest line are set down the means among all the former means, or numbers in each column, the numbers in which last line of means

means are found by adding into one sum the numbers in each column, and dividing that sum by the number of those parts. And thus we have obtained the mediums of the mediums for each day, which must be very near the truth. But to find how near they are to the truth, and to correct them, let these be collected and ranged as in the second column of the following table of the heights of charges, or column of irregular

Wt oz	Irregular		Regular	
	means	diffs	diffs	means
2	1.82	1.33	1.27	1.85
4	3.15	1.23	1.27	3.12
6	4.38	1.28	1.27	4.39
8	5.66	1.29	1.27	5.66
10	6.95	1.32	1.27	6.93
12	8.27	1.24	1.27	8.20
14	9.51	1.48	1.27	9.47
16	10.99	1.23	1.27	10.74
18	12.22	1.01	1.27	12.01
20	13.23			13.28

means. Then take the differences between each of these, and place them in the 3d column, or irregular differences; which would have been all equal if the mediums had been regular. Find then a medium among these unequal differences, by dividing their sum by the number of them, and it will be found to be 1.27, which set in the 4th column of regular or equal differences. Then, as the numbers in the 3d column, the nearest to this mean 1.27, are the differences between 6, 8, and 10 ounces, by supposing 5.66 to be the true length of the 8 ounce charge, I form all the others from it, by adding and subtracting continually the mean or common difference 1.27, and place them in the last column; which will therefore consist of the true regular length of each charge, including both the powder and the neck of the flannel bag which contained it.

How much of each space was really occupied by the neck of the bag, may be thus found: the first number 1.85 is the length of the charge

charge of 2 ounces, including the neck; and the common difference 1.27 is the real length of 2 ounces of powder in the bore; therefore, subtracting this number from the former, the remainder 0.58 is the mean length of the bore which was occupied by the neck of the bag in every charge. And, therefore, taking this number from each of those in the last column, the remainders will shew the real length of bore occupied by the powder alone in each of the charges.

OF THE RECOIL WITHOUT BALLS.

105. NEXT let us consider the quantity of recoil, or extent of the vibration of each gun, for every charge of powder; and first without balls. Now as these recoils were measured sometimes to one radius, and sometimes to another, it will be proper to reduce them all to a common radius, as well as to a common weight of gun when it happens to vary in weight. In the first year's experiments, the radius was various, and the chords of recoil were always taken in inches; but in those of the second and third years, the radius was constantly 10 feet, or 120 inches, which was divided into 1000 equal parts, and the chords of vibration measured in thousandth parts of the radius, each part being $\frac{12}{1000}$ of an inch. It will therefore be convenient to reduce the recoils of the first year, to the same radius and parts as those of the other two years: which may be done as follows:

Let r = any radius of the first year in inches,

and c = a corresponding chord of recoil taken in inches and parts.

Then $r : 120 :: c : \frac{120c}{r}$ the chord corresponding to the radius 120, and measured in inches;

and $120 : 1000 :: \frac{120c}{r} : \frac{1000c}{r}$ the same chord as expressed in thousandth parts of 120 inches.

Hence then, to reduce any chord of recoil, in the first year, multiply it by 1000, and divide the product by its own radius in inches; so shall the

the quotient be the corresponding chord answering to the radius 120 inches, and expressed in thousandth parts of that radius.

106. By the foregoing rule then having reduced all the chords of recoil to the radius 10 feet, and denoted them in thousandth parts of that radius; the mediums of every day's experiments, collected and arranged, will be as below.

Table of Recoils without Balls.

Charge of Powder, oz		2	4	6	8	12	16
GUN n° 1		22	53	85	113	176	221
		21	53		119	165	215
		21	55		116		220
		23	54		110		
		23	55		108		
		22			128		
		22			127		
	mediums	22	54	85	117	171	219
GUN n° 2		23	52		123		236
		23	56		118		240
		24			124		
		23			124		
		24					
	mediums	23	54		122		238
GUN n° 3		22	57	93	123		247
		23	59		125		250
		23	58		125		259
		23	56				252
		23					
		23					
		24					
	mediums	25					
		23	57½	93	125		252
GUN n° 4		25	58		127		280
		24	58		131		255
		26	56				261
		24	59				
	mediums	25	58		129		265

Some

Some of these mediums have not the greatest degree of exactness that they are capable of, for want of a sufficient number of repetitions, or numbers to take the mediums of. However, by a very small and obvious correction, the more accurate mediums, for the most usual charges of 2, 4, 8, and 16 ounces of powder, may be fairly stated as follows :

Gun n ^o	Center of gravity inches	Vibrat.	Length of bore inches	Powder			
				2 oz	4 oz	8 oz	16 oz
				Recoils without Balls			
1	80.47	40.1	28.2	22	53	117	220
2	80.47	40.0	38.1	23	55	121	237
3	80.50	39.9	57.4	24	57	125	252
4	80.44	39.8	79.9	25	59	129	265

The recoils being estimated in parts of which the radius is 1000 : and the common weight of the gun, with its frame and leaden weights, being 917 lb ; also the distance of the center of gravity below the axis, and the number of vibrations per minute, as set down in the 2d and 3d columns of the tablet above.

107. From the view and consideration of these numbers, various observations easily arise. As first, that, by observing the four columns, or vertical rows, it appears that the recoil of the gun, and consequently the force of the powder upon it, always increases as the length of the gun increases, and that in a manner tolerably regular as far as the charge of 8 ounces ; but after that, the increase is faster : thus, between the shortest bore of 28 inches long, and the longest of 80 inches, the increase in the velocity of recoil with 2 ounces of powder is from 22 to 25, or about the $\frac{1}{8}$ part ; with 4 ounces of powder, it is from 53 to 59, or the $\frac{1}{9}$ part ; and with 8 ounces of powder, it is from 117 to 129, or the $\frac{1}{10}$ part ; but with 16 ounces of powder, the increase is from 220 to 265, or the $\frac{1}{5}$ part. And this increase of recoil is chiefly,
if

if not intirely, to be ascribed to the longer time the fluid of the inflamed powder acts upon the gun, in passing through the greater length of bore; at least as far as to the charge of 8 ounces: but the extraordinary increase in the case of 16 ounces, seems to be partly owing to that, and partly to some of the powder, in this high charge, being blown out unfired from the short gun. And from this circumstance I would infer, that the whole of the charge of 8 ounces, without ball, is fired before it issues from the mouth of the short gun, that is before the fluid expands through a space of $22\frac{1}{2}$ inches of bore. And hence, if the velocity of the fluid were known, we could assign the time within which all the powder is fired. If, for instance, the mean velocity of the fluid were only 5000 feet in a second, though it is probably much more, the time would be only about the 250th part of a second in which the 8 ounces would be all inflamed.

The foregoing are the rates of increase in the chord of recoil, or in the velocity of the gun, which is proportional to it. It must be remarked however that the increase in the *force* of the powder will be about double of that of the recoil, because the force is as the square of the velocity: so that, from the shortest gun to the longest, the increase in the force of the powder with 2, 4, or 8 ounces, is about $\frac{1}{4}$, or from 4 to 5; and with 16 ounces of powder, the force is almost as 2 to 3, or the increase almost one half of the less force.

108. Again, if we contemplate the numbers on each horizontal line, that is, the recoils of each gun separately, with the several charges of 2, 4, 8, and 16 ounces of powder, we shall find that, in each of them, the recoil increases from the beginning, to a certain part, in a greater ratio than the constant ratio, 2 to 1, of the powder increases; and afterwards in a less ratio than that of the powder. That the ratio of the recoils, in every gun, is greatest at first, or with the least charges of powder: that the ratio continually decreases as the charge increases: that the ratio, at first, is greatest with the shortest gun, and so gradually

lefs and lefs all the way to the longeſt : but that, however, the ratio in the ſhorter guns decreaſe faſter than in the longer ; and ſo as to come ſooner to the ratio of 2 to 1 in the ſhorter guns than in the longer ; and after that, the ratios in the ſhort guns, with the ſame charge, are lefs than in the long ones. All theſe properties will perhaps appear ſtill plainer by arranging together the ſeveral ratios for each n° of gun, as here below :

Powder	Ratios for the Gun			
	n° 1	n° 2	n° 3	n° 4
2	2.41	2.39	2.37½	2.36
4	2.21	2.20	2.19	2.18
8	1.88	1.96	2.02	2.06
16				
means	2.17	2.18	2.19	2.20

where each column of ratios is found by dividing the recoils ſucceſſively by each other, from the beginning, namely, the recoil of 4oz by that of 2, the recoil of 8oz by that of 4, and the recoil of 16oz by that of 8. Alſo the firſt and ſecond lines rather decreaſe, but the 3d rather increaſes, and the laſt, or that of means, alſo rather increaſes.

And if we divide the firſt ratios, in the laſt table but one, ſucceſſively by each other, the 2d by the 1ſt, and the 3d by the 2d ; and then again theſe laſt ratios or quotients by each other ; and ſo on ; we ſhall obtain the ſeveral orders of ratios for each gun, as follows, obſerving uniform laws :

N° 1		N° 2		N° 3		N° 4	
22		23		24		25	
	2.41		2.39		2.37½		2.36
53	.917	55	.920	57	.922	59	.924
	2.21		2.20		2.19		2.18
117	.850	121	.891	125	.922	129	.945
	1.88		1.96		2.02		2.06
220		237		252		265	

where the firſt column, of each n° or gun, contains the recoils with

2, 4,

2, 4, 8, 16 ounces of powder; the 2d the first ratios, or the ratios of the recoils; the 3d contains the 2d ratios, or the ratios of the first ratios; and the last column contains the 3d ratios, or the ratios of the 2d ratios.

Or, perhaps, for some purposes it will serve better to set the same table in the following form, where the vertical columns are changed into horizontal lines :

N ^o 1	N ^o 2	N ^o 3	N ^o 4
22 53 117 220	23 55 121 237	24 57 125 252	25 59 129 265
2.41 2.21 1.89	2.39 2.20 1.96	2.37½ 2.19 2.02	2.36 2.18 2.06
.917 .850	.920 .891	.922 .922	.924 .945
.93	.97	1.00	1.02

OF THE RECOIL WITH BALLS.

109. BY collecting now the mean recoil of each gun for every day, after reducing them all to the same weight of gun, 917 lb, and weight of ball, 16 oz 13 dr, and to the same radius 1000, in the manner specified in Art. 105, they will stand as in this following table.

Powder, oz	2	4	6	8	10	12	14	16
GUN n° 1	90	148	197	241	260	290	294	329
	91	145	196	226	253	273	295	331
			199	234	260	281	313	331
				232		287		
				234				
				240				
				244				
				239				
mediums	90	146	197	236	258	283	301	330
GUN n° 2	92	152	207	249	274	296	305	360
	95	157		244	276	304	343	364
				244				348
				246				
mediums	94	154	207	246	275	300	324	358
GUN n° 3	99	166	216	259				399
	100	163	218	257				380
		164		260				
mediums	99	164	217	259				390
GUN n° 4	101	163		266				397
								417
mediums	101	163		266				407

Some

Some of these mediums are not very accurate, for want of a good number of repetitions, and especially those of the last gun n^o 4, which has only one duplicate. In this gun the recoils appear to be all rather low in respect of the others, but more especially that with the charge of 4 oz of powder, which is evidently much more defective than the rest, and requires an increase of about 6 to make it uniform with the others, and which increase it would probably have received from future experiments, had there been any repetitions of it. Augmenting therefore only that number by 6, all the orders of means will be tolerably regular, and stand as below, for the most usual charges of powder, namely, 2, 4, 8, and 16 ounces.

Gun n ^o	Powder			
	2	4	8	16
Recoils with Balls				
1	90	146	236	330
2	94	154	246	358
3	99	164	259	390
4	101	169	266	407

The common weight of ball being 16 oz 13 dr, and the weight of the gun 917 lb; the other circumstances being as in Art. 106.

110. From the several vertical columns of this tablet of means, we discover, that the recoils increase always as the length of the gun increases; but that in the 4th or longest gun, the increase is less, in proportion, than in the others. And from the horizontal lines we perceive, that the recoil always increases as the charge of powder increases, and that in a manner tolerably regular; and also in continued geometrical proportion when the charges of powder are so; but the common ratio in the former progression being only about $\frac{3}{4}$ of that in the latter. For, if the mediums, for each gun, be divided by each other, namely, the

2d by the 1st, the 3d by the 2d, and the 4th by the 3d, the quotients or ratios will come out as in the following tablet :

Powder	Ratios for the Gun			
	n° 1	n° 2	n° 3	n° 4
2	1.62	1.64	1.66	1.67
4	1.61	1.60	1.58	1.57
8	1.40	1.45	1.50	1.53
16				
means	1.54	1.56	1.58	1.59

where the numbers in the vertical columns, or the ratios for each gun separately, continually decrease ; and the numbers in the horizontal lines, or for the different guns with the same weights of powder, rather increase in the first and third line, but decrease in the second, and again rather increase in the last, which are the mediums of the three ratios in each column, and which mean ratios are rather more than $\frac{3}{4}$ of 2, the common ratio of the weights of powder, which are 2, 4, 8, 16 ounces.

And if we divide the numbers or ratios, in each column, continually by each other ; and their quotients by each other again ; the whole continued series or columns of ratios, for each gun, will be as here below :

N° 1	N° 2	N° 3	N° 4
90	94	99	101
1.62	1.64	1.66	1.67
146	154	164	169
1.61	1.60	1.58	1.57
.994	.976	.952	.940
.88	.93	1.00	1.04
236	246	259	266
.870	.906	.950	.974
1.40	1.45	1.50	1.53
330	358	390	407

where the first column, in each n° or gun, contains the recoils with 2,
5
4, 8,

4, 8, 16 ounces of powder; the 2d column contains the ratios of those recoils; the 3d contains the 2d ratios, and the last the 3d ratios.

Or the same table may, for some purposes, be more conveniently placed as below, where the vertical columns are ranged in horizontal lines:

N ^o 1	N ^o 2	N ^o 3	N ^o 4
90 146 236 330	94 154 246 358	99 164 259 390	101 169 266 407
1.62 1.61 1.40	1.64 1.60 1.45	1.66 1.58 1.50	1.67 1.57 1.53
.994 .870	.976 .906	.952 .950	.940 .974
.88	.93	1.00	1.04

OF THE MEAN VELOCITY OF THE BALL FROM THE RECOIL OF
THE GUN.

III. HAVING determined the mean recoil of the guns, both with and without balls, for the charges of 2, 4, 8, 16 ounces; we can now assign the mean velocity of the ball, for each gun and charge, from the recoils; if, as Robins has asserted, the force of the powder upon the gun be the same, whether it is fired with a ball or without one. For, if that property be generally true, then the velocity of the ball must be proportional to the difference of the chords of recoil with and without a ball; and that difference being multiplied by a certain constant number, will give the velocity of the ball itself; as we have before shewn.

Now if c denote the difference of those chords, b the weight of the ball, g the weight of the gun, g the distance to its center of gravity, i the distance to the axis of the bore, and n the number of oscillations the gun would make in a minute; then we have found, in Art. 68, that

$\frac{59}{96} \times \frac{g g c}{b i n}$ will express the velocity of the ball. And that when $G = 917$, $g = 80.47$, $i = 89.15$, and $n = 40$, which are the medium values of those letters, then the same theorem becomes $\frac{51}{4} \times \frac{c}{b}$ for the velocity of the ball. And, farther, when the mean value of b is 1.051 or 16 oz 13 dr, the same theorem for the velocity becomes barely $12\frac{1}{7}c$. Subtracting however the 700th part in the gun n° 1, and adding in the other three guns, as follows, namely,

the 1000th part in n° 2,

400th part in n° 3,

300th part in n° 4.

Therefore if each of the recoils without balls, in the last table of Art. 106, be taken from the corresponding recoils in Art. 109, and the remainders be multiplied by $12\frac{1}{7}$, making the additions and subtractions above-mentioned, we shall have the corresponding velocities of the ball by this method. And a synopsis of the whole, for each gun and charge, will be as in the following table :

Gun n°	Charges, 2 oz				4 oz				8 oz				16 oz			
	Recoil		Diff	Veloc of ball	Recoil		Diff	Veloc of ball	Recoil		Diff	Veloc of ball	Recoil		Diff	Veloc of ball
	with ball	without ball			with ball	without ball			with ball	without ball			with ball	without ball		
1	90	22	68	825	146	53	93	1127	236	117	119	1443	330	220	110	1334
2	94	23	71	863	154	55	99	1203	246	121	125	1520	358	237	121	1471
3	99	24	75	913	164	57	107	1302	259	125	134	1631	390	252	138	1680
4	101	25	76	926	169	59	110	1340	266	129	137	1669	407	265	142	1730

And we shall hereafter see how far these agree with the velocities computed from the vibration of the pendulum.

OF THE VELOCITY OF THE BALL, AS COMPUTED FROM THE
PENDULUM AND GUN.

112. THE four following tables contain the mediums of the velocities of the balls, as computed for each day, for all the principal charges of powder, and for each gun separately ; one table being allotted for each. In these tables all the mediums are arranged in a continued series, in the chronological order as they occurred, and accompanied with all the circumstances necessary to be known ; thus forming a fund or collection of elements, from which other arrangements and principles are to be deduced.

Each table consists of ten columns. The first column contains the dates ; the next three the state of the weather and air ; namely, the 2d column the hygrometer, or state of the air as to dryness and moisture ; the 3d the barometer ; and the 4th the thermometer ; both which last instruments, it must be observed, were always placed in the shade, and within the house, while the experiments were made in the open air, where it was commonly much hotter than the degree shewn by the thermometer. The 5th column contains the weight of the charge of powder ; the 6th and 7th the weight and diameter of the ball ; the 8th and 9th the velocity of the ball, the former as computed from the vibration of the pendulum, and the latter from the recoil of the gun ; and finally the 10th column contains the difference between these two velocities, which is marked with the negative sign (—) when the velocity by the gun is the less of the two.

Daily Mediums of Experiments with the Gun n° 1.

Date	Hygrometer	Barometer	Thermom	Powder	Ball's			Velocity by the		Diff
					wt		diam	pend	gun	
1783		inches	degrees	oz	oz	dr	inches	feet	feet	feet
June 30	dry	30.34	74	16	16	13	1.95	1456	1315	-141
July 17	dry	30.23	72	8	.	.	1.96	1471	1501	30
19	{	dry	30.12	70	2	.	.	797	832	35
					4	.	.	1109	1145	36
31	{	dry	30.13	69	12	.	.	1412	1374	-38
					16	.	.	1367	1334	-33
Aug 12	wet	30.00	64	16	.	12½	.	1419	1399	-20
Sept 10	{	dry	29.7	60	2	.	.	785	838	53
					4	.	.	1087	1122	35
					8	.	.	1353	1396	43
					8	.	13	1383		
					10	.	.	1417		
					12	.	.	1375		
					14	.	.	1333		
					16	.	.	1243D		
					20	.	.	1144		
					24	.	.	1194		
18	{	dry	30.08	64	32	.	.	880		
					36	.	.	838		
					6	.	14	1331		
					8	.	.	1386		
					10	.	.	1402		
					12	.	.	1453		
30	{	dry	30.25	64	14	.	.	1402		
1784										
Aug 4	wet			6	.	.	.	1295	1339	44
11	{	hazy	30.25	65	6	.	.	1368		
					8	.	15	1.97	1475	
					10	.	15	.	1493	
					12	.	14½	.	1520	
					14	.	14½	.	1528	
Sept 10	{	fair			2	.	12	1.96	755	
					4	.	.	1131	1170	39
					6	.	.	1370	1358	-12
					8	.	.	1475		
21	{	fair			4	.	12	1.97	1124	
					6	.	12	1.97	1372	
					8	.	11	1.96	1445	
Oct. 4	{	dry			2	.	13	1.96	759	
					4	.	12	1.96	1086	
					6	.	12	1.96	1325	
					8	.	12	1.96	1472	
					8	.	13	1.96	1411	
5	dry			8	.	9	1.95	1436		
6	dry			8	.					
11	hazy			8	.	7	1.95	1444		

Daily Mediums of Experiments with the Gun n° 2.

Date	Hygrometer	Barometer	Thermom	Powder	Ball's			Velocity by the		Diff
					wt	dr	diam	pend	gun	
1783		inches	degrees	oz	oz	dr	inches	feet	feet	feet
July 23	dry	29.88	70	2	16	13 $\frac{1}{2}$	1.96	793	840	47
				4	.	13 $\frac{1}{2}$	.	1135 ^D	1207	72
				8	.	13	.	1566	1592	26
				16	.	13	.	1660	1499	-161
Aug 12	wet	30.00	64	16	.	12 $\frac{1}{2}$	.	1676	1497	-179
Sept 11	dry	29.93	60	2	.	.	.	856	846	-10
				4	.	.	.	1239	1220	-19
				8	.	.	.	1571	1452	-119
				8	.	12	.	1569		
25	dry	29.93	59	10	.	.	.	1608		
				12	.	.	.	1615		
				14	.	.	.	1517 ^D		
				16	.	.	.	1664 ^D		
29	dry	30.28	64	6	.	11 $\frac{1}{2}$	.	1448		
				8	.	.	.	1561		
				10	.	.	.	1618		
				12	.	.	.	1669		
				14	.	.	.	1662		
				16	.	.	.	1637		
				18	.	11	.	1598		
				20	.	.	.	1639 ^D		
1785										
Sept 2	cloudy			8	.	13	1.96	1503		
9	dry			4	.	12	1.96	1204		

Daily Mediums of Experiments with the Gun n° 3.

Date	Hygrometer	Barometer	Thermom	Powder	Ball's			Velocity by the		Diff
					wt		diam	pend	gun	
1783		inches	degrees	oz	oz	dr	inches	feet	feet	feet
July 12	dry			16	16	13	1.96	2030	1706	-324
18	dry	30.28	68	4	.	.	1.96	1353	1321	-32
				8	.	.	.	1766	1620	-146
19	dry	30.12	70	2	.	.	.	898	921	23
Aug 13	cloudy	30.17	64	8	.	12½	.	1803	1594	-209
				16	.	.	.	1966	1542	-424
Sept 8	moist	30.03	61	2	.	13	.	926	928	2
				4	.	.	.	1334	1266	-68
1784										
Aug 5	dry	29.98	68	6	.	14	1.97	1616		
				4	15	2	1.87	1225		
				4	16	2	1.92	1244		
Sept 11	dry			4	16	14	1.97	1346		
				8	15	2	1.87	1662		
				8	16	3	1.92	1728		
				8	16	14	1.97	1815		
16				4	16	9	1.96	1388D		

Daily Mediums of Experiments with the Gun n° 4.

Date	Hygrometer	Barometer	Thermom	Powder	Ball's			Velocity by the		Diff
					wt		diam	pend	gun	
1783		inches	degrees	oz	oz	dr	inches	feet	feet	feet
July 29	dry	29.90	72	8	16	13	1.96	1936	1643	-293
				16	.	.	.	2161	1656	-505
30	dry	30.06	69	2	.	.	.	968	929	-39
				4	.	12	.	1375	1295	-80
1784										
Oct 12	dry			16	.	11	.	2060		

113. The foregoing tables contain the several mediums of velocities, for each day, and for all varieties in the circumstances of powder, and weight and diameter of ball. It will now then be proper to collect together all the repetitions of the same charge or weight of powder, and to take the mediums of all those mediums, to serve as fixed radical numbers, or established degrees of velocity, adapted to all the various charges of powder, and length of gun. And for this purpose, I shall reduce the numbers of these tables all to one common weight and diameter of ball, namely, to the weight 16 oz 13 dr, and the diameter 1.96 inches, which are the numbers that most commonly occur. And this reduction will be very well deduced from the experiments of September 11, 1784, when several trials were made with divers weights and diameters of ball, and with both 4 oz and 8 oz of powder, the results of which accord very well together. In the experiments of that day, it was found that, with the 4 oz charges, $\frac{1}{7}$ of the whole velocity is lost by the difference of $\frac{1}{16}$ of an inch in the diameter of the ball; and, with the 8 oz charge, $\frac{2}{3}$ of the velocity is lost by the same difference of windage. But the quantity of inflamed fluid which escapes, will be nearly as the difference between the area of the circle of the bore and the great circle of the ball, or the force will be as the square of the ball's diameter; and the velocity, we know, is as the square root of the force: and therefore the velocity is as the diameter of the ball; and the difference in the velocity, as the difference of the diameter, or as the windage. Hence, if w denote any difference of windage in parts of an inch, or difference between 1.96 and the diameter of any ball, and $\frac{1}{m}$ the part of the experimented velocity lost by $\frac{1}{16}$ of an inch difference of windage; then shall $\frac{1}{16} : w :: \frac{1}{m} : \frac{10w}{m}$, which last term will shew what part of the experimented velocity is lost by the increase of windage denoted by 10. By this rule then, I reduce all the velocities to what they would have been, had the diameter of the ball been always 1.96. It is to be noted however,

however, that the value of m will vary with the charge of powder: with 4 ounces of powder, it was found that $\frac{1}{m}$ was $\frac{1}{7}$ of the whole velocity, or $\frac{1}{6}$ of the experimented velocity; but with 8 oz of powder, $\frac{1}{m}$ was found to be $\frac{2}{13}$ of the whole, or $\frac{2}{13}$ of the experimented velocity. We shall not be far from the truth therefore, if we take the following values of $\frac{1}{m}$, to the several corresponding charges of powder; that is, as far as 16 oz in the guns n° 3 and 4, and then returning backwards again as the powder is increased above 16 oz, by 2 oz at a time; but in the gun n° 2, to continue only to 14 oz, and then return backwards again for all above 14 oz; and for the gun n° 1, to continue only to 12 oz, and then return backwards for all above that charge.

Powder	Value of $\frac{1}{m}$		
2	$\frac{2}{11}$	=	·182
4	$\frac{1}{6}$	=	·167
6	$\frac{4}{23}$	=	·160
8	$\frac{2}{13}$	=	·154
10	$\frac{4}{27}$	=	·148
12	$\frac{1}{7}$	=	·143
14	$\frac{4}{29}$	=	·138
16	$\frac{2}{13}$	=	·133

Such then is the reduction of the velocity on account of the windage. And as to that for the different weights of the ball, we know that the velocity varies in the reciprocal subduplicate ratio of the weight; and according to this rule the numbers were corrected on account of the different weights of ball. After these reductions then are made, the numbers in the foregoing tables, arranged under their respective charges of powder, will be as here below, for a ball of 1·96 diameter, and weighing 16 oz 13 dr.

Mean Velocities of Balls, for all the Guns with several Charges of Powder, reduced to a Ball of 1.96 Diameter, and weighing 16oz 13dr.



Powder, oz	2	4	6	8	10	12	14	16
GUN n° 1	797	1109	1334	1471	1417	1412	1333	1478
	784	1086	1298	1352	1405	1375	1405	1367
	754	1129	1371	1383	1476	1453	1511	1418
	759	1103	1368	1389		1503		1243D
		1084	1347	1458				
			1322	1472				
				1439				
				1469				
				1411				
				1447				
mediums	774	1102	1340	1431	1433	1436	1416	1377
GUN n° 2	794D	1136D	1444	1566	1605	1612	1657	1660
	855	1238		1569	1613	1664		1674
		1204		1566				1661
				1557				1632
				1503				
mediums	825	1191	1444	1552	1609	1638	1657	1656
GUN n° 3	898	1353	1593	1766				2030
	926	1334		1801				1966
		1327		1793				
		1378D						
mediums	912	1348D	1593	1787				1998
GUN n° 4	968	1373		1936				2161
								2052
mediums	968	1373		1936				2106

114. These last medium velocities, for each gun, will be tolerably near

near the truth; and the more so, commonly, as the number of the other mediums is the greater. For want, however, of a sufficient number of each sort, there are some small irregularities among the final mediums, which may be corrected, for the most part, by adding or subtracting 3 or 4 feet, as they are sometimes too little, and sometimes too great. And these small deviations will be very easily discovered by dividing the mediums by each other, namely, each of the velocities for 4, 6, 8, &c. ounces of powder, by that for 2 ounces. For we know, from the principles of forces, and other experiments, that the velocities will be nearly as the square roots of the quantities of powder; that is, while the length of the charge does not much shorten the length of the bore before the ball; but gradually deviating from that proportion more and more, as the charge of powder is increased in length; because the force has gradually a less distance and time to act upon the ball in. Now by dividing the quantities of powder 4, 6, 8, &c. by 2, the quotients 2, 3, 4, &c. shew the ratios of the charges; and the roots of these numbers, namely,

1.414

1.732

2.000

&c.

shew the ratios which the velocities would have to each other nearly, if the empty part of the bore was always of the same length. But as the vacant part always decreases as the charge increases, the ratios of the velocities may be expected to fall short of those above, and the sooner and the more so, as the gun is shorter. Accordingly, on trial, we find the ratios hold pretty well, even in the shortest gun, as far as to the 6 oz charge; but in the 8 oz charge it falls about $\frac{1}{13}$ or $\frac{1}{14}$ part below the true ratio, being 1.85 instead of 2. In the longer guns, the proportions hold out gradually longer, and the deviations are always less and less: thus, in the 2d gun, the ratio for the 8 oz charge is about 1.895, in the 3d it is 1.945, and in the 4th gun it is 1.999 or 2 very nearly. And so for other charges. Correcting then some of the mediums by
means

means of this property, the more accurate radical medium velocities, for each gun, with the several charges of 2, 4, 6, and 8 ounces of powder, will be as here below :

Powder	Gun n° 1			N° 2			N° 3			N° 4		
	Ratio.	Veloc.	Dif 1. 11.	Ratio.	Veloc.	Dif 1. 11.	Ratio.	Veloc.	Dif 1. 11.	Ratio.	Veloc.	Dif 1. 11.
2		780			835			920			970	
4	1'140	1100	320	1'414	1180	345	1'413	1300	380	1'412	1370	400
6	1'731	1340	240	1'730	1445	265	1'729	1590	290	1'732	1680	310
8	1'850	1430	90	1'893	1580	135	1'945	1790	200	2'000	1940	260

where the velocity is set in large characters in the middle column; on the left hand in a small character, is the ratio, which is found by dividing each velocity by the first, the law of which ratios has been mentioned above; and on the right hand are the columns of first and second differences; the first being the difference between each two succeeding numbers, and the second the differences of those differences.

Or, for some purposes, it may be more convenient to range the velocities, &c. as here below :

Gun n°	2 oz			4 oz			6 oz			8 oz		
1	780			1'410	1100		1'731	1340		1'850	1430	
2	835	55		1'414	1180	80	1'730	1445	105	1'893	1580	150
3	920	85		1'413	1300	120	1'729	1590	145	1'945	1790	210
4	970	50		1'412	1370	70	1'732	1680	90	2'000	1940	150

where the numbers are here placed in horizontal lines, which before were vertical; and vertical here, those which before were horizontal. And where the law, both of the ratios and differences, is evident. We also hence perceive how, for each charge, the velocity of the ball is continually increased as the gun is longer.

And these velocities may be considered as standard radical numbers, here deposited, and ready to be applied to any purpose, in which the consideration of the velocity can be useful. And those for the other charges of powder will be as in the general table in Art. 113.

115. These velocities however, it must be remarked, are those with which the ball strikes the pendulum, after passing through the air between it and the muzzle of the gun; and consequently they are less than the velocities with which it immediately issues from the gun, by as much velocity as the ball loses by the resistance of the air, in its flight through that space. Now we have found, in Art. 33, that the first velocities lose at least their 84th part by that resistance, when the air behind the ball is supposed instantly to fill up the place always quitted by the ball in its flight. But as this is not exactly the case, the air rushing into a vacuum with a certain finite velocity, therefore the part lost will be gradually more and more as the ball moves swifter, till its velocity become equal to that of the air itself; after which the part lost will remain constant. And Mr. Robins asserts that the velocity lost by very swift motions, is about 3 times as great as that lost by slow ones; and therefore that will be about the 28th part. So that the loss will always lie between the 84th part and the 28th part. I shall therefore leave it in this uncertain state, till other experiments enable us to ascertain what may be the exact proportion of loss peculiar to every degree of velocity.

116. From the general table of medium velocities in Art. 113, it is evident that, for each gun, the velocity increases with the charge to a certain extent, where it is greatest; and that afterwards it gradually decreases as the charge is increased. It farther appears that the point, or charge, at which the velocity is the greatest, is different in the guns of different lengths; the charge which gives the maximum of velocity, being always greater, as the gun is longer. And by tracing this increase of charge, from the beginning, to the point of greatest velocity, it appears that, with the 1st, 2d, and 3d guns, the charges which give the greatest velocities, are nearly as follows, viz.

Gun n° 1	at the charge of	120z,
- - 2	- - - - -	140z,
- - 3	- - - - -	160z.

Here

Here it will not be so proper to specify what portion of the weight of the ball these weights of powder are; being no ways regulated by that circumstance; but what portion of the bore of the gun is filled with these quantities of powder. Now, by the table of the lengths of charges in Art. 104, it appears that the lengths of the charges of 12, 14, and 16 oz, are these, viz.

12 oz	- -	8.20 inches; gun 1, its length 28.2
14 oz	- -	9.5 inches; gun 2 - - - 38.1
16 oz	- -	10.7 inches; gun 3 - - - 57.4

Then dividing each length of charge by its corresponding length of gun, we obtain nearly these three following fractions, viz.

$$\begin{aligned} \frac{3}{15} & \text{ in gun 1 of 15 calibers long} \\ \frac{1}{4} & \text{ in gun 2 of 20 calibers long} \\ \frac{3}{16} & \text{ in gun 3 of 30 calibers long} \end{aligned}$$

which expresses what part of the bore is filled with powder, when the greatest velocity is given to the ball, with each of these lengths of gun. And which therefore is not one and the same constant part for all lengths of gun, but varying nearly in the reciprocal subduplicate ratio of the length of the bore.

117. Having so far settled the degree of velocity of the ball, as determined by the vibration of the pendulum, we may in like manner now proceed to assign the mean velocities, as deduced from the recoil of the gun. The repetitions in this latter way are not so numerous as in the former; but, such as they are, we shall here abstract them from the general tables in Art. 112, reducing them, however, all to the same common weight and diameter of ball, as was done in Art. 113.

Mean Velocities from the Recoil of the Gun.

Powder, oz	2	4	6	8	12	16
GUN n° 1	832	1145	1344	1501	1374	1337
	837	1120	1352	1393		1334
		1165				1396
	mediums	835	1143	1348	1447	1374
GUN n° 2	841	1209		1592		1499
	845	1218		1450		1494
	mediums	843	1213		1521	1496
GUN n° 3	921	1321		1620		1706
	928	1266		1591		1540
	mediums	925	1294		1605	1623
GUN n° 4	929	1293		1643		1656

These mediums however are not so exact as those in Art. 111, because those were deduced from a greater number of particulars. We shall therefore chiefly adopt those that were stated in that article, for the radical standard velocities of the ball, as determined from the recoil of the gun, excepting in some instances when the other is used, and sometimes the mediums of both. So that the final mediums will be as follows :

Velocities of the Ball from the Recoil of the Gun.

Gun n°	2 oz	4 oz	8 oz	16 oz
1	830	1135	1445	1345
2	863	1203	1521	1485
3	919	1294	1631	1680
4	929	1317	1669	1730

118. Let us now compare these velocities, deduced from the recoil of the gun, with those that are stated in Art. 113 and 114, which were determined from the pendulum; that we may see how near they will agree together. And, in this comparison, it will be sufficient to employ the velocities for 2, 4, 8, and 16 ounces of powder; this will be the most certain also, as these mediums are better determined than most of the others.

Comparison of the Velocities by the Gun and Pendulum.

Gun n ^o	2 oz			4 oz			8 oz			16 oz		
	Velocity by		Dif	Velocity by		Dif	Velocity by		Dif	Velocity by		Dif
	gun	pend		gun	pend		gun	pend		gun	pend	
1	830	780	50	1135	1100	35	1445	1430	15	1345	1377	-32
2	863	835	28	1203	1180	23	1521	1580	-59	1485	1656	-171
3	919	920	-1	1294	1300	-6	1631	1790	-159	1680	1998	-318
4	929	970	-41	1317	1370	-53	1669	1940	-271	1730	2106	-376

In this table, the first column shews the number of the gun; and its velocity of ball, both by the vibration of the gun and pendulum, with their differences, is on the same line with it, for the several charges of powder. After the first column, the rest of the page is divided into four spaces, for the four charges, 2, 4, 8, 16 ounces; and each of these is divided into three columns: in the first of the three, is the velocity of the ball as determined from the vibration of the gun; in the second is the velocity as determined from the vibration of the pendulum; and in the third is the difference between the two, which is marked with the negative sign, or —, when the former velocity is less than the latter, otherwise it is positive.

119. From the comparison contained in the last article, it appears, in general, that the velocities, determined by the two different ways, do not agree

agree together; and that therefore the method of determining the velocity of the ball from the recoil of the gun, is not generally true, although Mr. Robins and Mr. Thompson had suspected it to be so; and consequently that the effect of the inflamed powder on the recoil of the gun, is not exactly the same when it is fired without a ball, as when it is fired with one. It also appears that this difference is no ways regular, neither in the different guns with the same charge, nor in the same gun with different charges of powder. That with very small charges, the velocity by the gun is greater than that by the pendulum; but that the latter always gains upon the former, and soon becomes equal to it; and then exceeds it more and more as the charge of powder is increased. That the particular charge, at which the two velocities become equal, is different in the different guns; and that this charge is less, or the equality sooner takes place, as the gun is longer. And all this, whether we use the actual velocity with which the ball strikes the pendulum, or the same increased by the velocity lost by the resistance of the air, in its flight from the gun to the pendulum.

OF THE RANGES AND TIMES OF FLIGHT.

120. HAVING dispatched what relates to the velocity of the ball, we may now proceed in like manner to the experiments made to determine the actual ranges, and the times of flight of the balls.

The mediums of these, hitherto obtained, are not so numerous as could be wished; however, such as they are, we shall here collect them in the same manner as we did the circumstances relating to the initial velocities in Art. 112.

Mediums

Mediums of Ranges and Times of Flight.

Date	Hygrometer	Barometer	Thermom	Powder	Ball's			Elevat gun	Time ft	Range
					wt		diam			
1785		inches	degrees	oz	oz	dr	inches	degrees	fec	feet
Sept 2	cloudy	29.80	66	8	16	13	1.965	15	14.0	5916
8	hazy	30.02	65	8	16	12	1.96	15	14.7	6216
14	clear	30.50	67	4	16	12	1.96	15	8.4	4398
28	{	30.35	60	4	16	10	1.963	15	8.3	4523
				2	16	8	1.97	45	22.0	5068
29	{	30.35	60	2	16	12	1.95	45	20.3	5150
				12	16	12	1.95	15	15.5	6700
1786										
June 12	{	29.89	63	4	16	3	1.957	15	11.0	5060
				2	16	5	1.959	15	9.2	4130
1785										
Oct 4	rain	29.93		8	15	3	1.96	15		5600
11	clear	29.88	60	8	15	12	1.96	15	10.1	5620

Of these, the first 6 days experiments were with the gun n° 2 ; and the last two days, with the gun n° 3.

121. Now, by taking again the mediums of these, both in the balls, and their ranges and times of flight, they will finally come out as follows :

Final Mediums of Ranges and Times.

GUN	Powder	Ball's		Elevat gun	Time ft	Range	Veloc. ball
		wt	diam				
	oz	oz	dr	inches	degrees	sec	feet
N° 2	2	16	10	1.96	45	21.2	863
	2	16	5	1.959	15	9.2	868
	4	16	8 $\frac{1}{3}$	1.96	15	9.2	1234
	8	16	12 $\frac{1}{2}$	1.962	15	14.4	1644
	12	16	12	1.95	15	15.5	1676
N° 3	8	15	7 $\frac{1}{2}$	1.96	15	10.1	1938

And in the last column are added the corresponding initial velocities, which the ball would have at the muzzle of the gun; which have been extracted from the medium velocities, as determined by the pendulum, and here reduced to the peculiar weight and diameter of ball in each particular case of this table, by the reductions specified in Art. 113, and by augmenting the velocity for the 2 ounce charge by its 36th part, and the others by their 28th part, for the loss of velocity in passing from the gun to the pendulum.

So that in this little table, we have the following concomitant data, determined with a tolerable degree of precision; namely, the weight of powder, the weight and diameter of the ball, the initial or projectile velocity, the elevation of the gun, the time of the ball's flight, and its range, or the distance on the horizontal plane. From which it is hoped that the resistance of the medium, and its effect on other elevations, &c. may be determined, and so afford the means of deriving easy rules for the several cases of practical gunnery: a subject intended to be prosecuted in a future volume of these Tracts.

OF THE BALL'S PENETRATION INTO THE WOOD.

I SHALL here select only the depths of the penetrations into the block of wood, that have been made in the course of the last year's experiments, as they are the most numerous and uniform, and were all made with the same gun, namely, n° 2. I shall also select only those for 2, 4, and 8 ounces of powder, as they are the most useful and certain numbers for affording safe and general conclusions; and besides, the trials with other charges are too few in number, being commonly no more than one of each.

Mean

Mean Penetrations of Balls into Elm Wood.

Powder	2	4	8
	7	16.6	18.9
		13.5	21.2
			18.1
			20.8
			20.5
means	7	15	20

That is, the balls penetrated about

7 inches deep with 2 oz of powder

15 - - - - 4 - - - -

20 - - - - 8 - - - -

And these penetrations are nearly as the numbers

2, 4, 6, or 1, 2, 3; but the quantities of powder are

as 2, 4, 8, or 1, 2, 4; so that the penetrations are as the charges as far as 4 ounces, but in a less ratio at 8 ounces, namely, less in the ratio of 3 to 4. And are indeed, so far, proportional to the logarithms of the charges.

Now, by the theory of penetrations, the depths ought to be as the charges, or, which is the same thing, as the squares of the velocities. But from our experiments it appears that the penetrations fall short of that proportion in the higher charges. And therefore it would seem, that the resisting force of the wood is not uniformly the same; but that it increases a little with the increased velocity of the ball. And this probably may be occasioned by the greater quantity of fibres driven before the ball; which may thus increase the spring or resistance of the wood, and so prevent the ball from penetrating so deep as it otherwise would do. But it will require farther experiments in future to determine this point more accurately.

124. Before we conclude this tract, it may not be unuseful to make a short recapitulation of the more remarkable deductions that have been drawn from the experiments, in the course of these calculations. For by bringing them together into one collected point of view, we may, at any time, easily see what useful points of knowledge are hereby obtained, and thence be able to judge what remains yet to be done by future experiments. Having therefore experimented and examined all the objects that were pointed out in art. 5, p. 104, & *seq.* we shall just slightly mention the answers to these enquiries; which are either additions to, or confirmations of, those laid down p. 102, as drawn from our former experiments in the year 1775.

And 1st, then, it may be remarked that the former law, between the charge and velocity of ball, is again confirmed, namely, that the velocity is directly as the square root of the weight of powder, as far as to about the charge of 8 ounces: and so it would continue for all charges, were the guns of an indefinite length. But as the length of the charge is increased, and bears a more considerable proportion to the length of the bore, the velocity falls the more short of that proportion.

2nd. That the velocity of the ball increases with the charge, to a certain point, which is peculiar to each gun, where it is greatest; and that by further increasing the charge, the velocity gradually diminishes till the bore is quite full of powder. That this charge for the greatest velocity is greater as the gun is longer, but not greater however in so high a proportion as the length of the gun is; so that the part of the bore filled with powder bears a less proportion to the whole in the long guns, than it does in the shorter ones; the part of the whole which is filled being indeed nearly in the reciprocal subduplicate ratio of the length of the empty part. And the other circumstances are as in this

Table of Charges producing the Greatest Velocity.

Gun n ^o	Length of the bore	Length filled	Part of the whole	Weight of the powder
	inches	inches		oz
1	28.2	8.2	$\frac{3}{10}$	12
2	38.1	9.5	$\frac{3}{12}$	14
3	57.4	10.7	$\frac{3}{15}$	16
4	79.9	12.1	$\frac{3}{20}$	18

3dly. It appears that the velocity continually increases as the gun is longer, though the increase in velocity is but very small in respect to the increase in length, the velocities being in a ratio somewhat less than that of the square roots of the length of the bore, but somewhat greater than that of the cube roots of the length, and is indeed nearly in the middle ratio between the two. But the particular degrees of velocity for each gun, and charge, may be seen at p. 255 and 257.

4thly. It appears, from the table of ranges in art. 121, p. 263, that the range increases in a much less ratio than the velocity, and indeed is nearly as the square root of the velocity, the gun and elevation being the same. And when this is compared with the property of the velocity and length of gun in the foregoing paragraph, we perceive that we gain extremely little in the range by a great increase in the length of the gun, the charge being the same. And indeed the range is nearly as the 5th root of the length of the bore; which is so small an increase, as to amount only to about $\frac{1}{7}$ th part more range for a double length of gun.

5thly. From the same table in art. 121, it also appears that the time of flight is nearly as the range; the gun and elevation being the same.

6thly. It appears that there is no difference caused in the velocity or range, by varying the weight of the gun, nor by the use of wads, nor by different degrees of ramming, nor by firing the charge of powder in different parts of it.

7thly. But a very great difference in the velocity arises from a small degree of windage. Indeed with the usual established windage only, namely, about $\frac{1}{20}$ th of the caliber, no less than between $\frac{1}{3}$ and $\frac{1}{4}$ of the powder escapes and is lost. And as the balls are often smaller than that size, it frequently happens that $\frac{1}{2}$ the powder is lost by unnecessary windage.

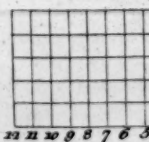
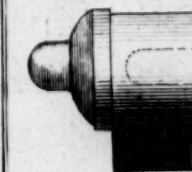
8thly. It appears that the resisting force of wood, to balls fired into it, is not constant. And that the depths penetrated by different velocities or charges, are nearly as the logarithms of the charges, instead of being as the charges themselves, or, which is the same thing, as the square of the velocity.

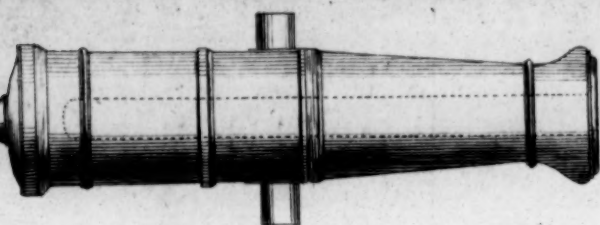
9thly. These, and most other experiments, shew that balls are greatly deflected from the direction they are projected in; and that so much as 300 or 400 yards in a range of a mile, or almost $\frac{1}{4}$ th of the range, which is nearly a deflection of an angle of 15 degrees.

10thly. Finally, these experiments furnish us with the following concomitant data, to a tolerable degree of accuracy; namely, the dimensions and elevation of the gun, the weight and dimensions of the powder and shot, with the range and time of flight, and first velocity of the ball; from which it is to be hoped that the measure of the resistance of the air to projectiles may be determined, and thereby lay the foundation for a true and practical system of gunnery, which may be as well useful in service as in theory; especially after a few more accurate ranges are determined with better balls than some of the last employed on the foregoing ranges.

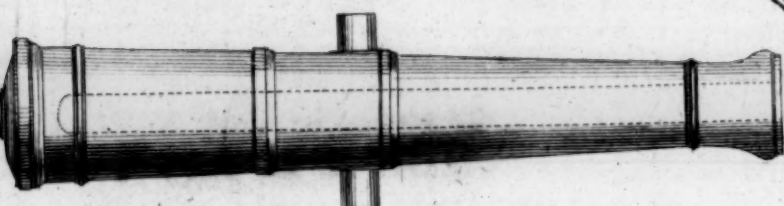


ADDITIONS

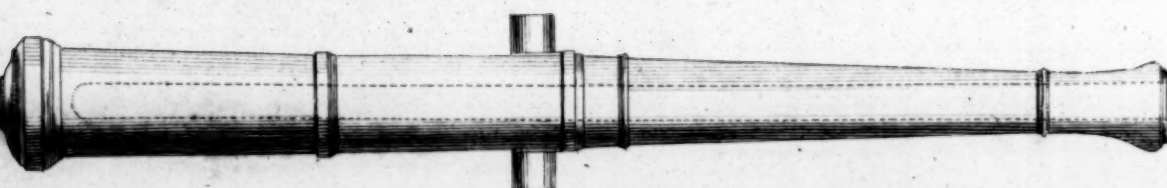




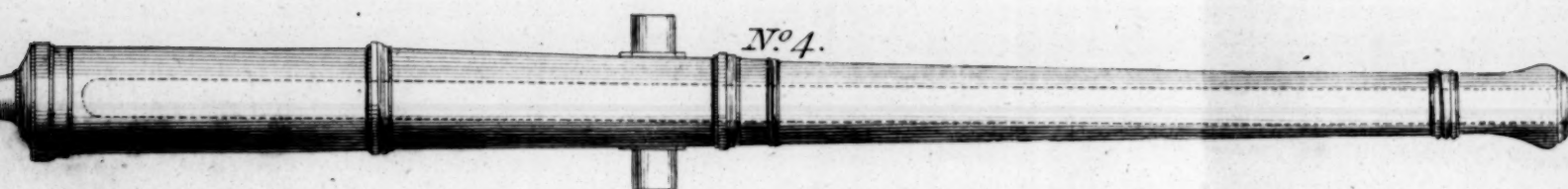
N^o 1.



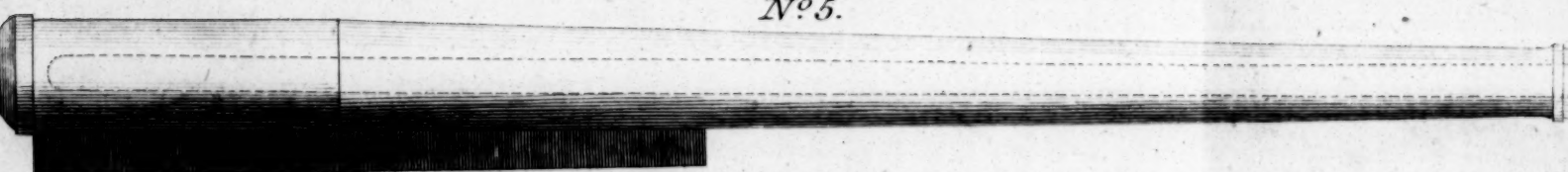
N^o 2.



N^o 3.

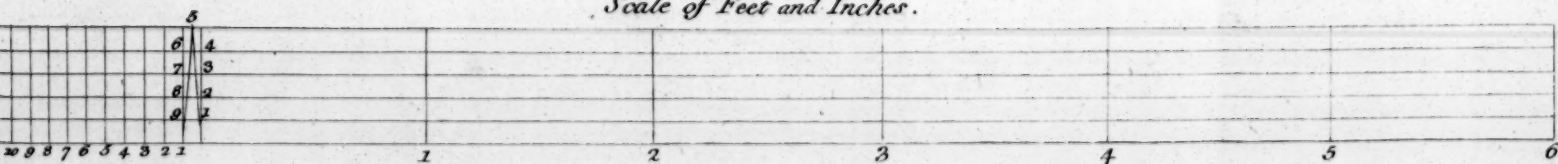


N^o 4.



N^o 5.

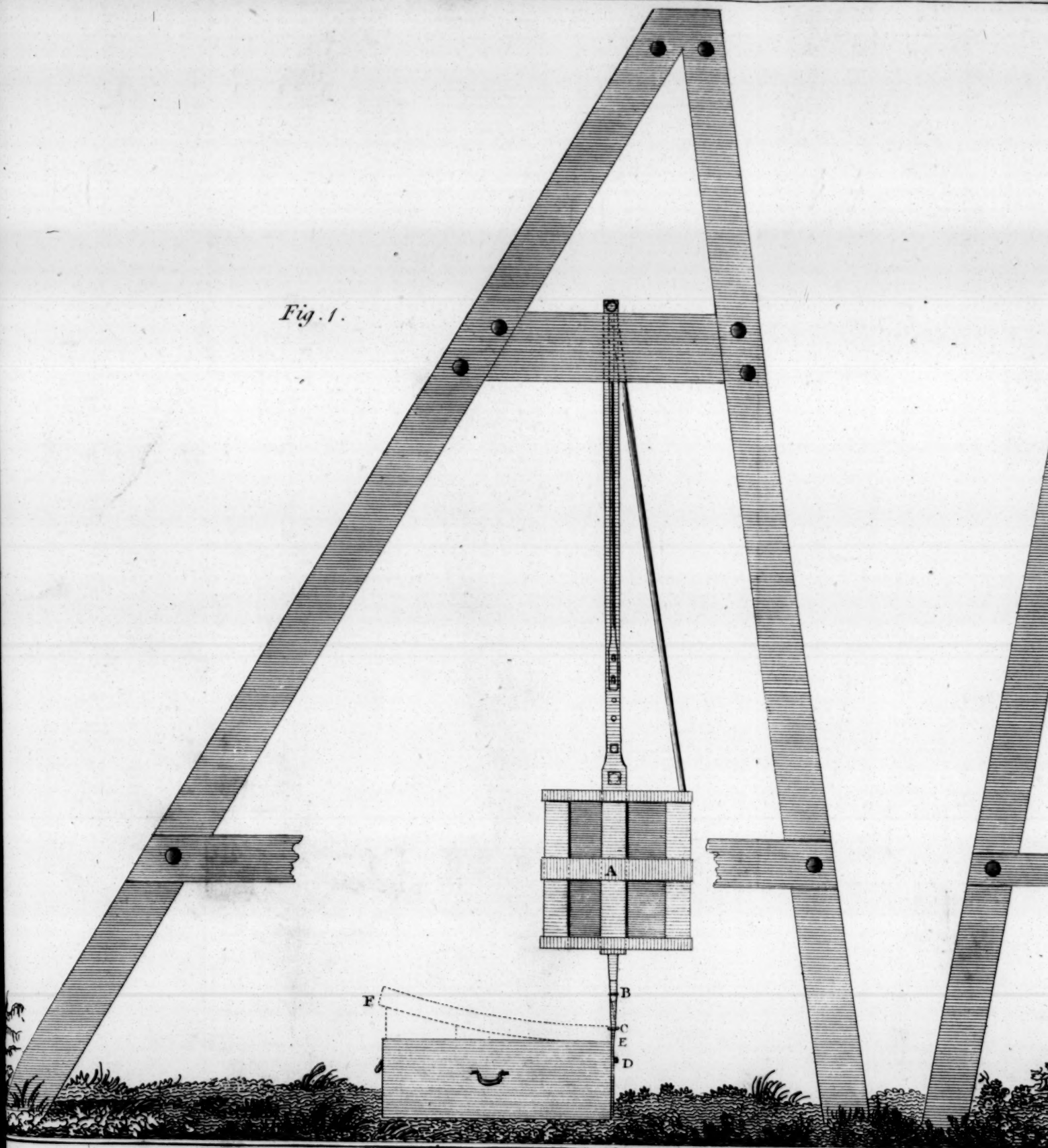
Scale of Feet and Inches.



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Fig. 1.



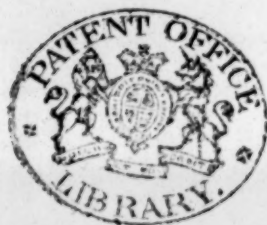


Fig. 2.

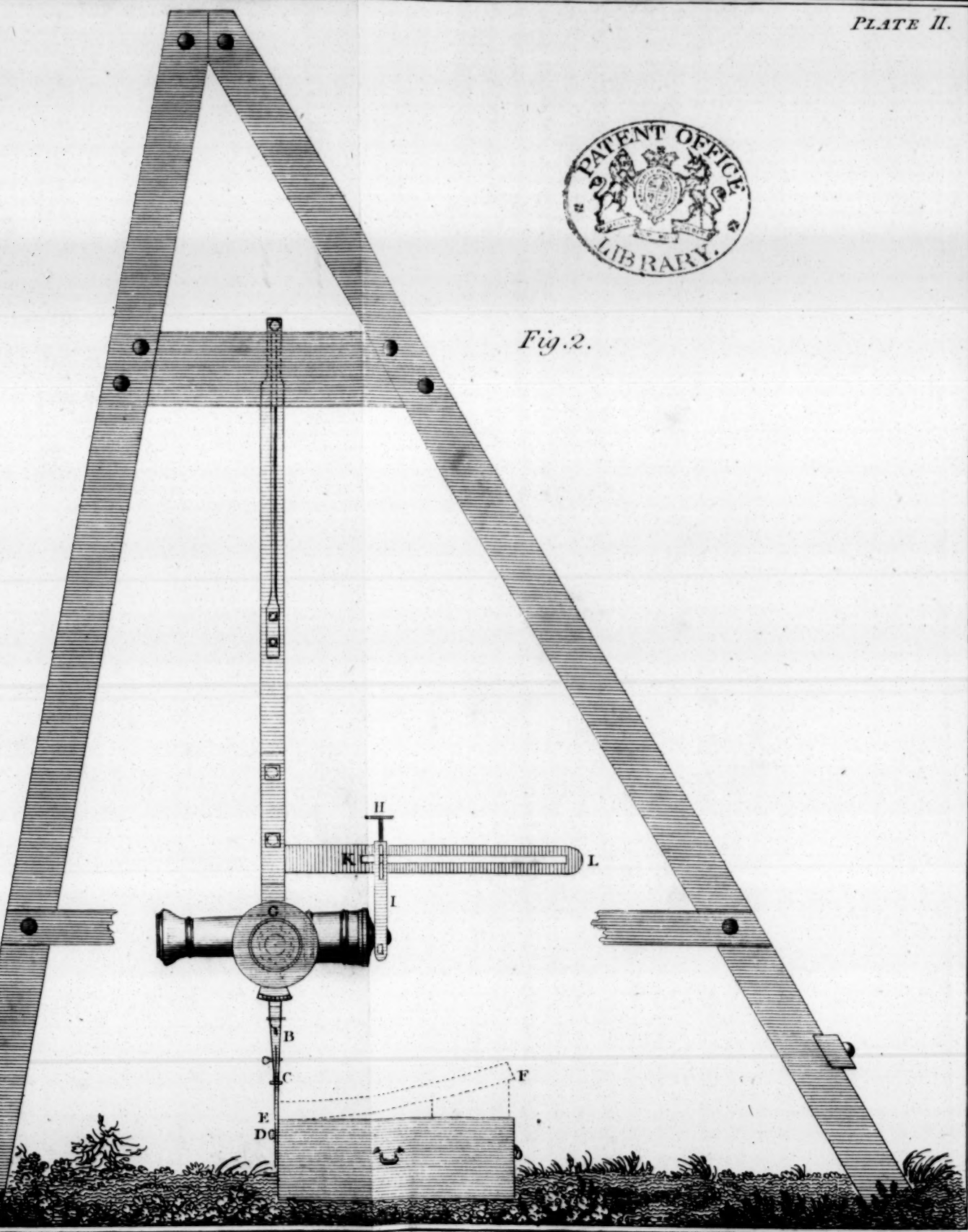


Fig. 1.

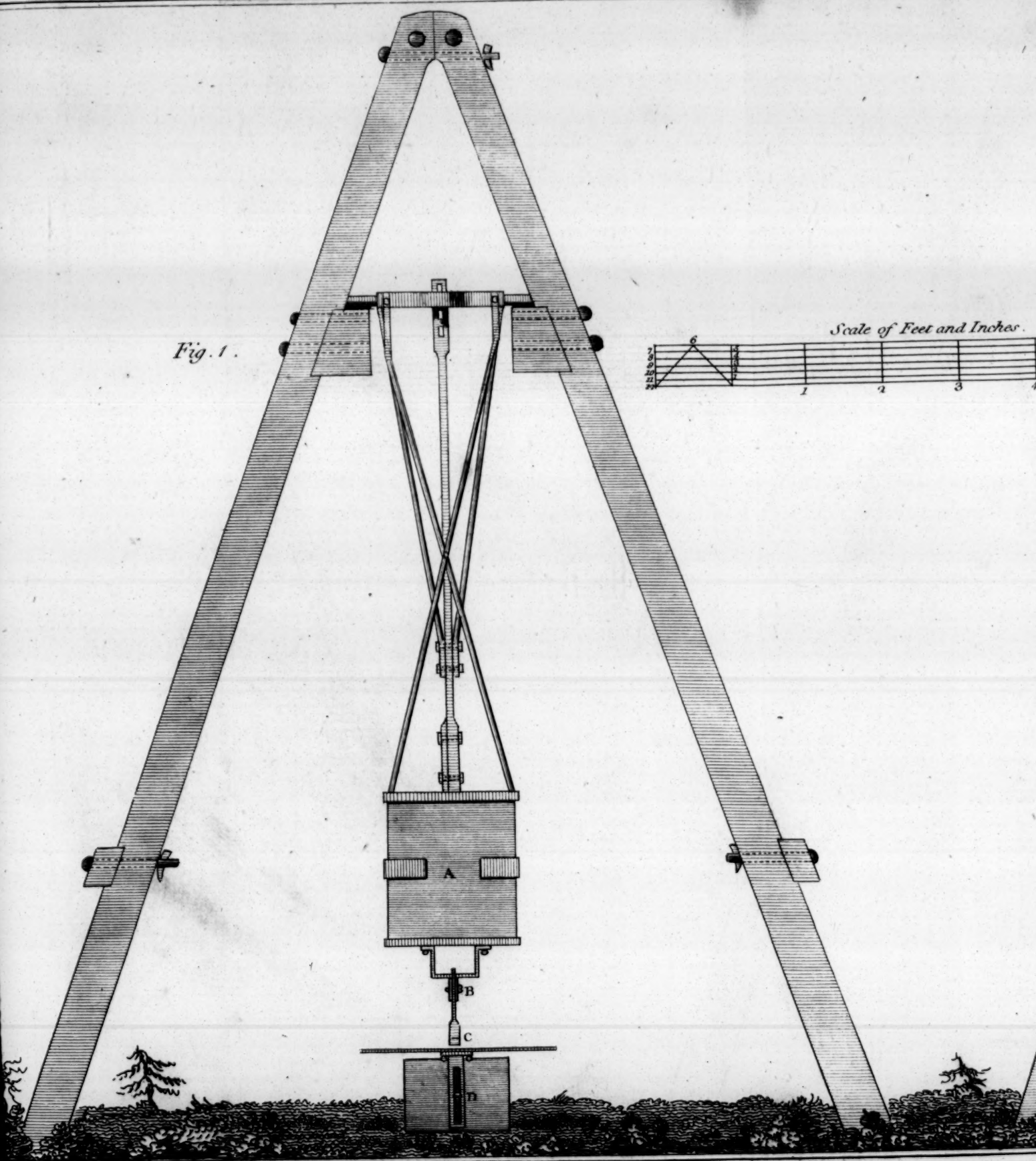
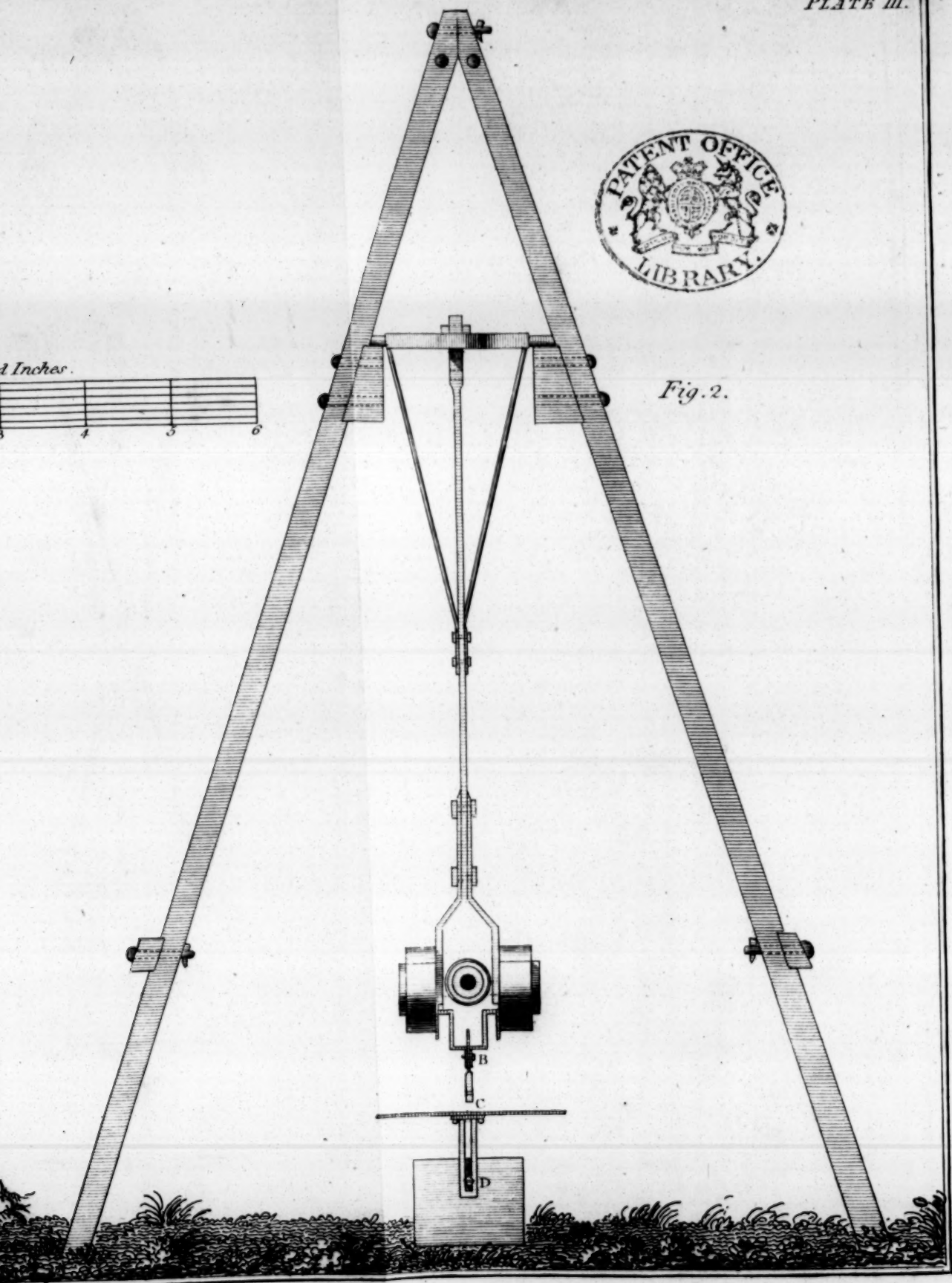
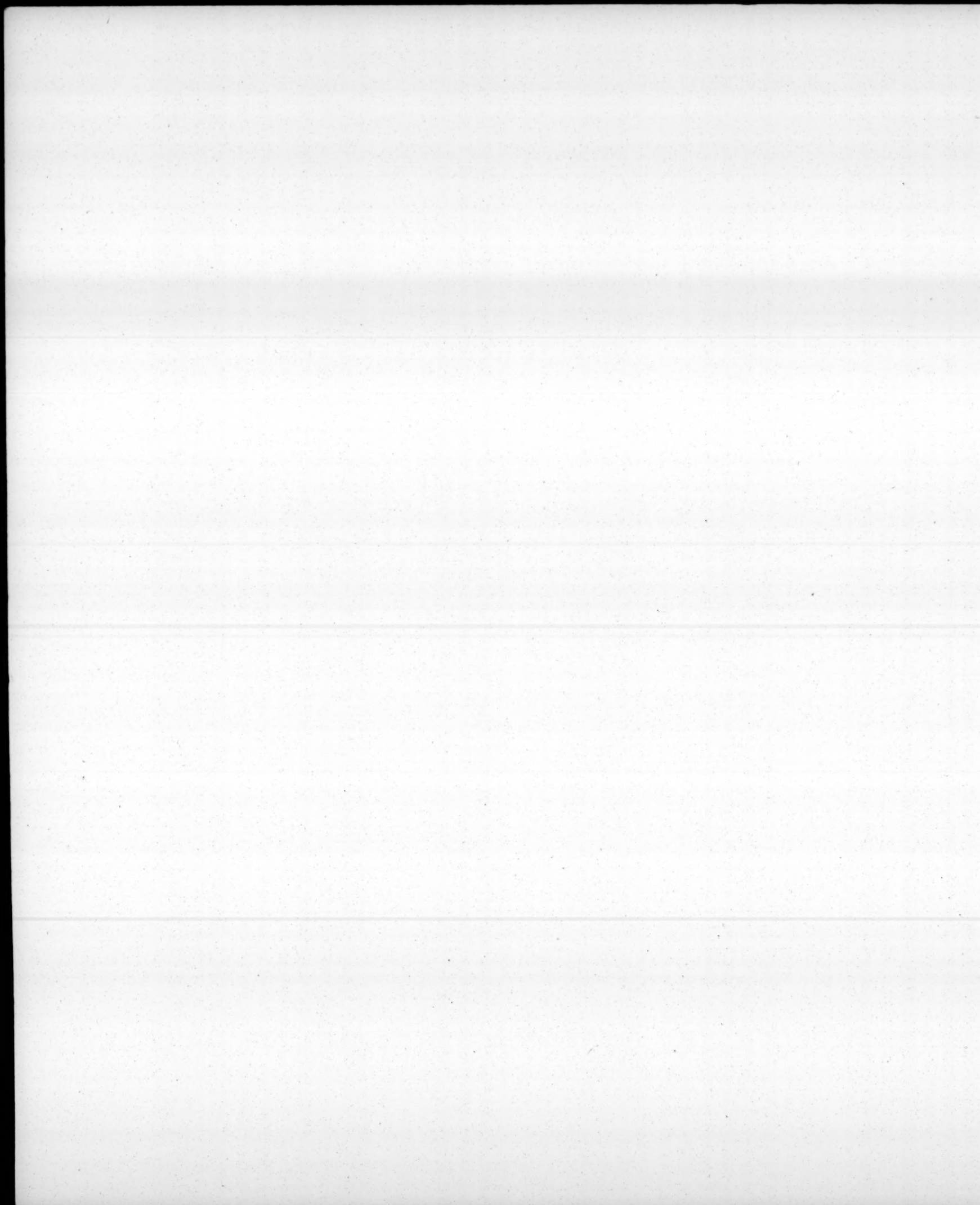
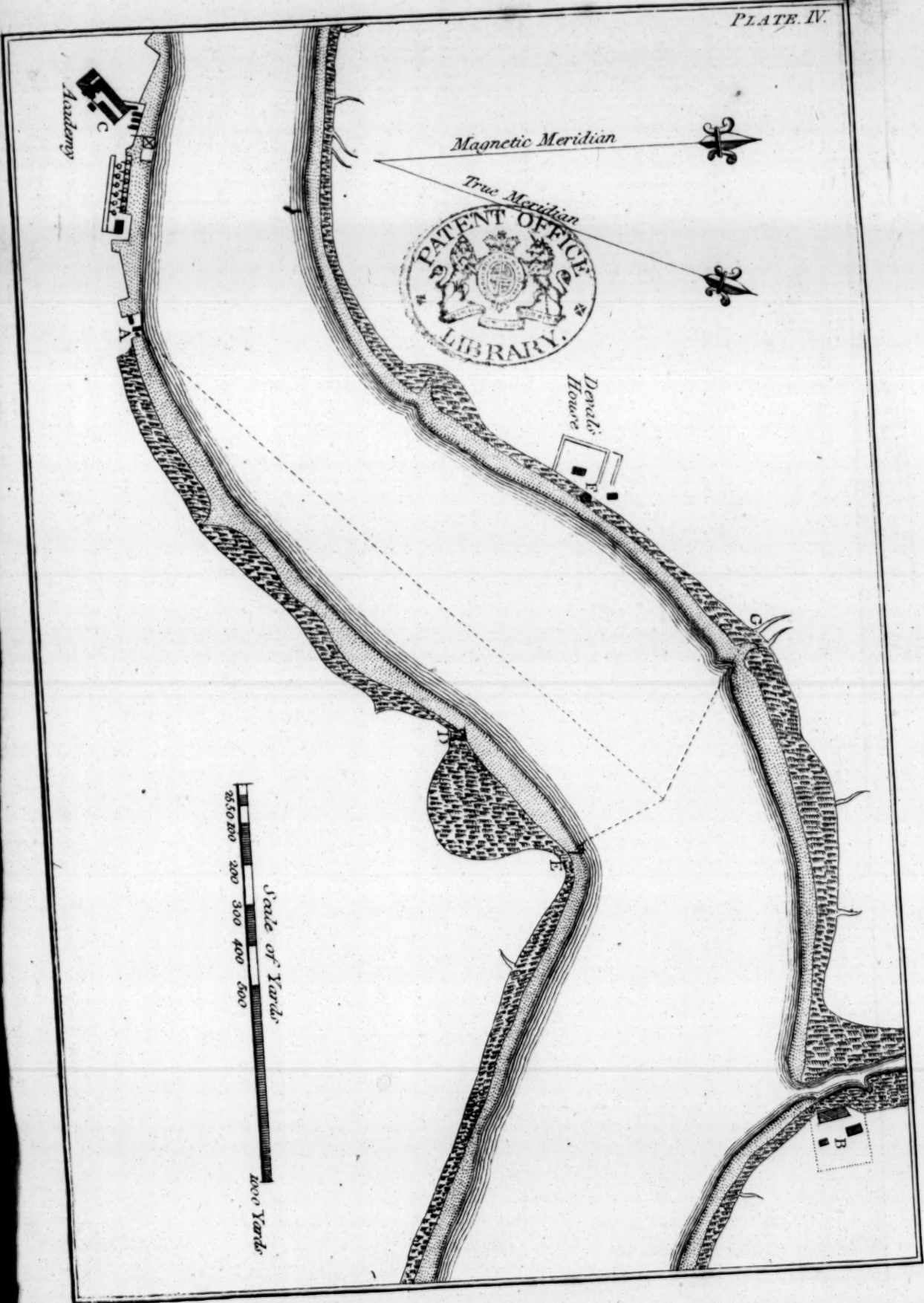




Fig. 2.







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ADDITIONS AND CORRECTIONS.

Pa. 6, l. 17, for $-\frac{1}{2}$, read $\frac{1}{-2}$.

Pa. 12, l. 18, for operation, read operations.

Pa. 59, l. 7, for $3\sqrt{r}$, read $\sqrt[3]{r}$.

Pa. 259, l. 19, at the end of *Art.* 116, add as follows; or still nearer in the reciprocal subduplicate ratio of the empty part of the bore before the charge. And by this rule finding the part for the longest gun, or n° 4, it will be found to be $\frac{3}{20}$, or 12.1 inches in length, answering to 18 ounces of powder. So that the whole set of numbers, for the greatest velocity, will be as follows:

Gun n°	Length of the bore	The Charge		
		Wt oz	Length	
			Inches	Part of whole
1	28.2	12	8.2	$\frac{3}{15}$
2	38.1	14	9.5	$\frac{3}{12}$
3	57.4	16	10.7	$\frac{3}{10}$
4	79.9	18	12.1	$\frac{3}{20}$

F I N I S.

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